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A Direct Solution for the Two-Way Parabolic Equation

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ABSTRACT: The method of calculating the reflected field using the Parabolic Equation (PE) approximation described in Ref. [1] is based on an efficient iteration method. However, the iteration scheme does not converge for certain properties of the acoustic environment. A direct method for determining the reflected field has been established to overcome this convergence problem and is, therefore, applicable for a wider range of acoustic environments. Comparison of results from both the iteration and the direct method is shown to verify the direct method of calculating the reflected field. Furthermore, an example is shown where the iteration method fails for the reflected field, but in this case the direct method returns a solution to the problem. The result obtained by the direct method is compared to the result achieved by a two-way coupled-mode model, see Ref. [2].

1. INTRODUCTION

In underwater acoustics the reflected field and, therefore, the reverberation caused by variations in the acoustic environment are significant for certain acoustic systems. Numerical modelling of the sound propagation in the ocean gives the possibility to identify and explain the properties of the reverberation sources. In modelling of large scale sound propagation, the reflected field or reverberation from deterministic features may be obtained by solving the two-way wave equation, e.g. solving the wave equation for the outgoing and incoming acoustic field. Especially, the deterministic variations in the ocean bottom can be analyzed by two-way numerical models, where the changes in the bathymetry are described by a finite discretization of the bottom.

Several approaches have been published to solve this kind of problems based on for example PE, Normal Modes, Finite Element Methods and Boundary Element Methods, see Ref. [1], [2], [3] and [4]. However, the models based on the PE approximation of the wave equation are known to be efficient for range dependent environments, and applicable for wide-angle sound propagation. The two-way PE model developed by M. D. Collins *et al*, see Ref. [1], is capable to handle both the forward and reflected field efficiently and at very wide propagation angles. This PE model is based on a single-scattering approach, where the reflected field is obtained by utilizing an iteration method. The iteration scheme fails for certain acoustic environments with moderate impedance changes and small stair-steps for describing variations in the bathymetry. These problems may be solved by applying a direct method for computing the reflected field instead of the iteration scheme in the two-way PE approximation.

The direct formulation of the reflected field from changes in the bathymetry has been established and implemented in the PE model. The reflected field is determined by a discretization of the variations in the bathymetry using a finite number of stair-steps and solving the boundary conditions at each vertical interface. In order to assess the validity of the direct formulation, an analysis of two test examples for the reflected field has been performed. In the first test example the reference solution is obtained by the PE model based on the iteration method for calculating the reflected field. In the second example the iteration method does not converge, but the direct method returns a solution to the problem. In this case the coupled-mode model developed by R. B. Evans, see Ref. [2], is applied for generating the reference solution.

2. FORMULATION OF THE DIRECT PE METHOD

The derivation of the reflected field in the PE approximation considered is described in Ref. [1]. A brief introduction of the derivation will be given in the following resulting in the direct formulation of the reflected field. To derive the outgoing and incoming field the two dimensional reduced and far-field wave equation for range independent regions is considered. The wave equation is given by:

$$\frac{\partial^2 \Psi}{\partial x^2} + \rho \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial \Psi}{\partial z}\right) + k^2\,\Psi = 0 \tag{1}$$

where x is the horizontal distance from a line source assuming plane geometry, z is the depth coordinate below the sea surface, Ψ is the complex acoustic field and ρ is the density. In case of cylindrical geometry the x in Eq. 1 is the horizontal distance from a point source, and the spreading factor $x^{-\frac{1}{2}}$ has to be added the complex acoustic field. The complex wave number k in Eq. 1 is defined as:

$$k = \left(1 + i\,\frac{\beta}{40\,\pi\,\log_{10}\,e}\right)\frac{\omega}{c}, \tag{2}$$

where ω is the angular frequency, c is the sound speed and β is the attenuation in dB per wavelength. As Eq. 1 is written for range independent environments, the sound speed and the density are only functions of the depth. In this case the wave equation factors, and the terms which must be satisfied by the outgoing and incoming field can be identified. The equation which has to be fulfilled by the acoustic field is, therefore, given by:

$$\left(\frac{\partial}{\partial x} + i\,k_0\sqrt{1+X}\right)\left(\frac{\partial}{\partial x} - i\,k_0\sqrt{1+X}\right)\Psi = 0 \tag{3}$$

where X is defined as:

$$X = k_0^{-2} \left(\rho \frac{\partial}{\partial z} \frac{1}{\rho} \frac{\partial}{\partial z} + k^2 - k_0^2 \right) \tag{4}$$

and k_0 is the reference wavenumber. The first term of Eq. 3 involving the term $+ik_0$ must be satisfied by the incoming field and the second term, which includes the $-ik_0$, has to be fulfilled by the outgoing field. The square root in Eq. 3 may be approximated by the following rational-linear function:

$$L = \sqrt{1 + X} \cong 1 + \sum_{j=1}^{n} \frac{\alpha_j X}{1 + \beta_j X} \tag{5}$$

where α_j and β_j are complex Padé coefficients tabulated in Ref. [5] for the sum approximation of the depth operator. By applying the sum approximation, the outgoing and incoming field must satisfy:

$$\frac{\partial \Psi}{\partial x} = \pm i\, k_0 \left(1 + \sum_{j=1}^{n} \frac{\alpha_j X}{1 + \beta_j X} \right) \Psi \tag{6}$$

Range dependent environments are handled by a discretization of the environment into range independent segments forming a sequence of stair-steps. At the vertical interface between two range independent segments A and B, continuity of the acoustic field and the normal component of particle velocity is required. This is expressed as:

$$\begin{aligned} \Psi_i + \Psi_r &= \Psi_t \\ \frac{1}{\rho_A} \frac{\partial}{\partial x} (\Psi_i + \Psi_r) &= \frac{1}{\rho_B} \frac{\partial}{\partial x} (\Psi_t) \end{aligned} \tag{7}$$

where Ψ_i, Ψ_r and Ψ_t are the incident, reflected and transmitted field at the vertical interface, respectively, and ρ_A and ρ_B are the density in the two range independent segments A and B, respectively. Combining the two boundary conditions and utilizing the approximation given by Eq. 5 and 6, the following relation between the incident and reflected field is obtained:

$$\left(\frac{1}{\rho_A} L_A + \frac{1}{\rho_B} L_B \right) \Psi_r = \left(\frac{1}{\rho_A} L_A - \frac{1}{\rho_B} L_B \right) \Psi_i \tag{8}$$

The incident field and, therefore, the right hand side of Eq. 8 is known by solving for the outgoing field, see Eq. 6, and is in the following denoted by Ξ. The equation for determining the reflected field is now written as:

$$\left(\frac{1}{\rho_A}L_A + \frac{1}{\rho_B}L_B\right)\Psi_r = \Xi \tag{9}$$

which in general requires inversion of a large complex matrix given by L_A and L_B for each range independent segment A and B. To obtain a convenient procedure of solving Eq. 9 the depth operator L is written as a rational-linear product approximation as follows:

$$L = \sqrt{1+X} \cong \prod_{j=1}^{n} \frac{1+a_j X}{1+b_j X} = \frac{N}{D} \tag{10}$$

where the X is given by Eq. 4, and the complex Padé coefficients a_j and b_j for the product approximation are tabulated in Ref. [1]. The numerator N and denominator D of the depth operator L are general banded matrices, where the number of diagonals depends on the number of Padé coefficients in the product approximation. By applying the definition of N and D and utilizing the property that N and D commute, e.g. $N\,D^{-1}{=}D^{-1}\,N$, Eq. 9 is rewritten as:

$$\left(\frac{1}{\rho_A}D_A^{-1}N_A + \frac{1}{\rho_B}N_B D_B^{-1}\right)\Psi_r = \Xi \tag{11}$$

Rearranging Eq. 11 yields the following expression, see Ref. [6]:

$$\frac{1}{\rho_A}D_A^{-1}\left(N_A D_B + D_A\frac{\rho_A}{\rho_B}N_B\right)D_B^{-1}\,\Psi_r = \Xi \tag{12}$$

and by matrix operations the reflected field is determined by:

$$\Psi_r = D_B\left(N_A D_B + D_A\frac{\rho_A}{\rho_B}N_B\right)^{-1} D_A\rho_A\,\Xi \tag{13}$$

The reflected field is expressed explicitly and can hereby be calculated directly from Eq 13. The formulation does not include the iteration procedure described in Ref. [1] which in certain cases fails. The new formulation has been implemented and is verified in the next section by comparing results from the two-way PE based on the iteration method and a coupled-mode model.

3. NUMERICAL VERIFICATION

To verify the formulation of the direct solution of the reflected field in the PE model, test example B in Ref. [1] has been adopted. In this case there is a single step in the ocean bottom for evaluating the reflected field. The bathymetry is range independent for ranges less than 7 km with an ocean depth equal to 500 m. At the range from 7 km to 10 km the ocean depth is constant at 250 m. In the water column the sound speed is 1500 m/s, the density 1 g/cm^3 and the attenuation equal to 0 dB per wavelength. The acoustic parameters for the bottom have been divided into two examples. In example 1 the sound speed in the bottom is 1700 m/s, the density 1.5 g/cm^3 and the attenuation equal to 0.5 dB per wavelength. The acoustic parameters in example 2 are the same as in example 1 except for the sound speed which is chosen to 2400 m/s.

A 25 Hz line source is placed at a depth of 50 m below the sea surface assuming plane geometry. The number of Padé coefficients n in Eq. 5 for the PE approximation is 5 for the outgoing field and, at present, limited to 3 in Eq. 10 for the reflected field in both example 1 and 2. A convergent solution is obtained for both examples by choosing the depth grid spacing to 1 m and a range step of 5 m in the PE modelling. The presented results are obtained by running the models on an IBM 355 RISC SYSTEM/6000 using the maximum optimization level during the compilation.

The reflected field in terms of transmission loss in dB for example 1 using the PE model based on the iteration and direct method is shown in Fig. 1 and 2, respectively.

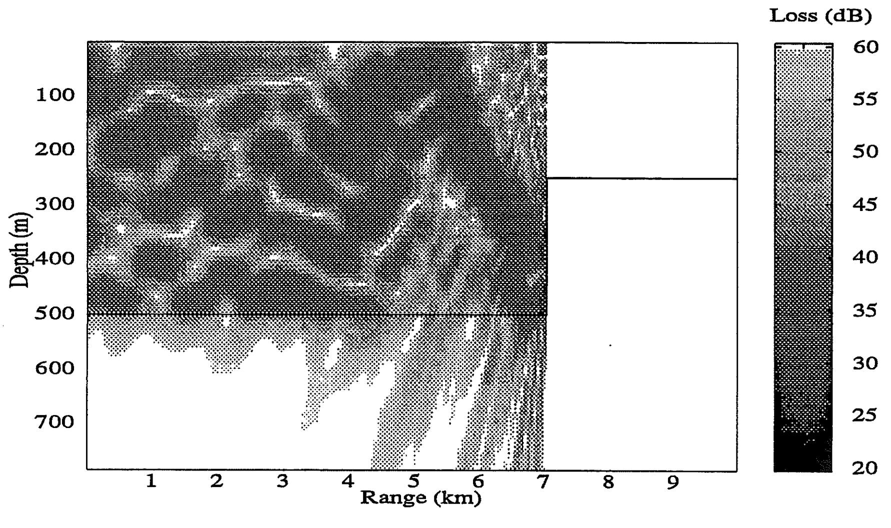

Figure 1: The reflected field in example 1 determined by using the PE model based on the iteration method.

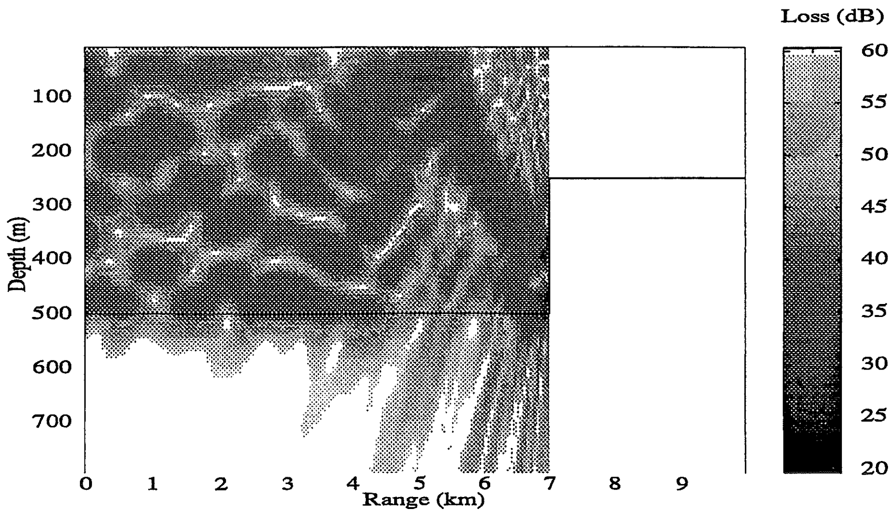

Figure 2: The reflected field in example 1 calculated by using the PE model based on the direct method.

The agreement between the results of the reflected field obtained by the PE model based on the iteration and the direct method is excellent. The implementation of direct method in the PE model is performed successfully, and the direct method handles the environment given in example 1 as the iteration method. The CPU time for the two-way analysis is 31 *sec* and 50 *sec* for the iteration and direct method, respectively.

In Fig. 3 and 4 the reflected field for example 2 in terms of the transmission loss in dB is shown calculated by using the direct PE model and the coupled-mode model, respectively.

In example 2 the iteration method for calculating the reflected field does not converge caused by the increased sound speed in the bottom. A fully convergent solution is obtained by including 50 modes and a total depth to the false bottom of 1500 m using the coupled-mode model.

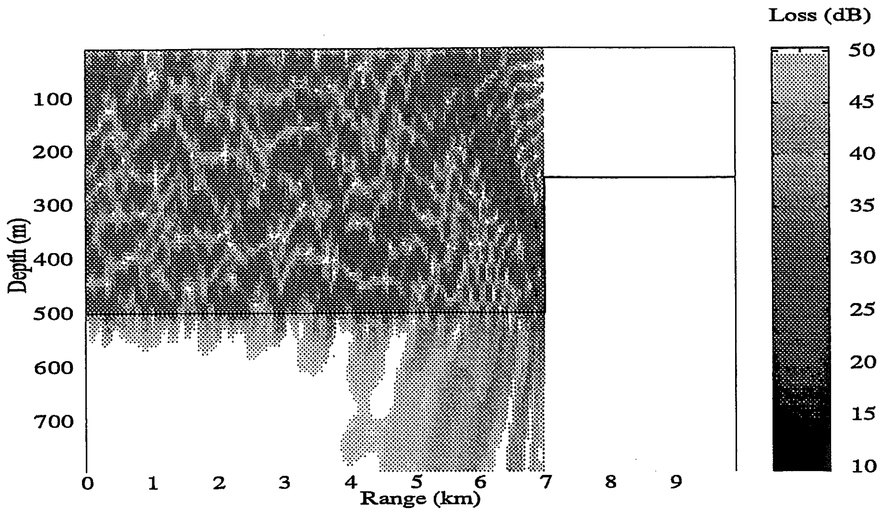

Figure 3: The reflected field for example 2 calculated by using the PE model based on the direct method.

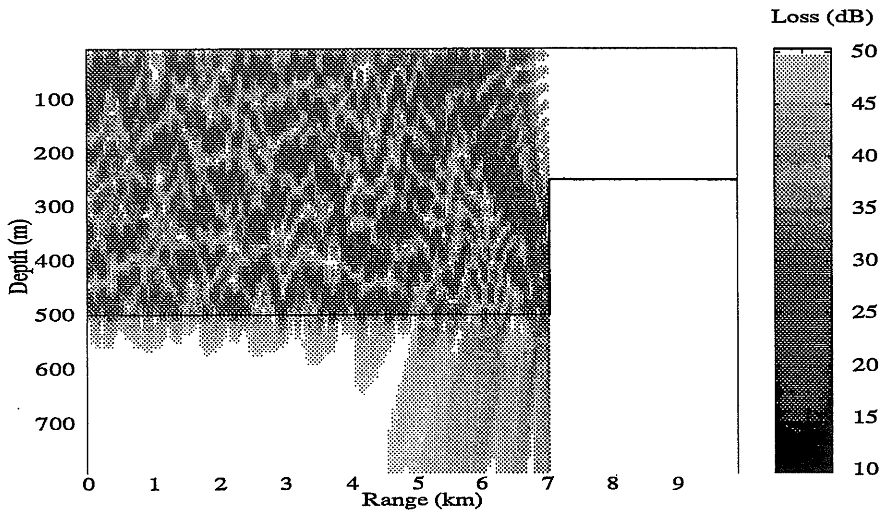

Figure 4: The reflected field for example 2 calculated by using the coupled-mode model.

There is consistency in the solution for the reflected field obtained by the PE model based on the direct method and the coupled-mode model. However, there is a slight lower loss for the direct PE model at the position of the step in the vertical direction. This small difference between the PE and the coupled-mode model is caused by the limited number of Padé coefficients in the scattering routine of the PE model. The CPU time for the analysis of example 2 is 50 *sec* and 19 *sec* for the PE and the coupled-mode model, respectively.

4. CONCLUSIONS

The method of determining the reflected field using the PE approximation has been improved by introducing the direct PE formulation. It has been shown that the direct method returns the same solution as the iteration method analyzing the same acoustic environment. However, if the environment has certain properties the iteration method for determining the reflected field fails, but the solution can be obtained by the PE approximation based on the direct formulation. In this case the reference solution is found by using a coupled-mode model. There is a slight difference between the result obtained by the direct PE and the coupled-mode model, which is caused by the limited number of Padé coefficients in the scattering routine of the direct PE model.

5. ACKNOWLEDGEMENTS

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References

[1] M. D. Collins and R. B. Evans, "A two-way parabolic equation for acoustic backscattering in the ocean", Journal of Acoustical Society of America **91** (3), March 1992, pp. 1357-1368.

[2] R. B. Evans, "A coupled mode solution for acoustic propagation in a waveguide with stepwise depth variations of a penetrable bottom", Journal of Acoustical Society of America **74** (1), July 1983, pp. 188-195.

[3] J. E. Murphy and S. A. Chin-Bing, "A finite-element model for ocean acoustic propagation and scattering", Journal of Acoustical Society of America **86** (4), October 1989, pp. 1478-1483.

[4] P. Gerstoft and H. Schmidt, "A boundary element approach to ocean seimoacoustic facet reverberation", Journal of Acoustical Society of America **89** (4), April 1991, pp. 1629-1642.

[5] M. D. Collins, "Higher-order Padé approximations for accurate and stable elastic parabolic equations with application to interface wave propagation", Journal of Acoustical Society of America **89** (3), March 1991, pp. 1050-1057.

[6] M. D. Collins, "Higher-Order, Energy-Conserving, Two-Way, and Elastic Parabolic Equations", PE Workshop II, Proceedings of the Second Parabolic Equation Workshop, Edited by Stanley A. Chin-Bing, David B. King, James A. Davis, and Richard B. Evans, May 1993, pp. 145-168.