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A Numerical Solution of Parabolic Elastic Wave Equations

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Abstract

Based on the parabolic equation approximation, a set of equations have been developed for three-dimensional time-harmonic wave propagation in an elastic medium. The elastic equations for the scalar and vector displacement potentials are written in a matrix form which is a direct counterpart to previous work on the scalar wave equation for a fluid medium. This paper introduces a numerical solution of the elastic equations. An ordinary differential equation (ODE) method in conjunction with a finite difference scheme leads to a stable marching procedure. One feature of this approach is that every finite-difference discretization results in a tridiagonal system of equations; these equations can be solved efficiently by recursive formulas. The criterion for choosing the range step size is discussed, and the stability and accuracy of the method are analyzed.

1 Introduction

For the treatment of coupled fluid-elastic acoustics problems, a combination of three models is needed: one to represent the wave propagation in the fluid medium, one to represent the wave propagation in the elastic medium, and one to represent the fluid-elastic interface conditions. A complete integration of these three models would constitute a solution to a general class of problems involving fluid-elastic interactions, including some interactions which are important in shallow water ocean environments. This paper discusses a partial solution to the fluid-elastic interaction problem – a numerical solution to a set of representative wave equations in an elastic medium.

This paper focuses on the three-dimensional, one-way propagation problem in an elastic medium. There exist a number of models for the numerical solution of one-way or parabolic elastic equations. Most of the previous computational methods are two-dimensional; a review of these developments can be found in ref. [3]. We attack here the zeroth-order problem of the model developed by Nagem et al. [3]. This model was purposely developed in such a way that is analagous to and compatible with existing three-dimensional one-way propagation models in the fluid medium. Without details, we first give the relevant elastic equations derived in ref. [3]. Next, the set of elastic equations are written in a vector ordinary differential equation (ODE) form. A numerical ODE solution is introduced to solve this vector ODE by means of single scalar equations with successive substitutions. The criterion for choosing the range step size is discussed, and stability and convergence of the numerical method are examined.

2 Three-Dimensional Elastic Equations

Nagem et al. [3] applied a perturbation method to derive equations for wave propagation in an inhomogeneous elastic medium. The formulation is three-dimensional, with arbitrary variations in elastic constants and in density. The elastic displacement vector $\mathbf{u}^{(0)}$ is written as

$$\mathbf{u}^{(0)} = \nabla\phi^{(0)} + \nabla \times \psi^{(0)}, \tag{2.1}$$

where $\phi^{(0)}$ and $\psi^{(0)}$ are the zeroth-order scalar and vector displacement potentials. The potentials are further written as

$$\phi^{(0)} = r^{-1/2}A^{(0)}(r,\theta,z)e^{ik_L r} \tag{2.2}$$

$$\psi_r^{(0)} = r^{-3/2}B_r^{(0)}(r,\theta,z)e^{ik_T r} \tag{2.3}$$

$$\psi_\theta^{(0)} = r^{-1/2}B_\theta^{(0)}(r,\theta,z)e^{ik_T r} \tag{2.4}$$

$$\psi_z^{(0)} = r^{-1/2}B_z^{(0)}(r,\theta,z)e^{ik_T r}, \tag{2.5}$$

where $(\psi_r, \psi_\theta, \psi_z)$ are the components of ψ with respect to the (r, θ, z) cylindrical coordinate system. A diagonalization technique is used to convert the equations for the envelope functions $A^{(0)}$, $B_r^{(0)}$, $B_\theta^{(0)}$ and $B_z^{(0)}$ into an operator form which distinguishes outgoing waves from incoming waves. After the separation, the zeroth-order outgoing wave equations are

$$\left(\frac{\partial}{\partial r} + ik_L^{(0)} - ik_L^{(0)}\sqrt{1+L_L}\right) A^{(0)} = 0 \tag{2.6}$$

$$\left(\frac{\partial}{\partial r} + ik_T^{(0)} - ik_T^{(0)}\sqrt{1+L_T}\right) B_r^{(0)} = 0 \tag{2.7}$$

$$\left(\frac{\partial}{\partial r} + ik_T^{(0)} - ik_T^{(0)}\sqrt{1+L_T}\right) B_\theta^{(0)} = \frac{1}{2ik_T^{(0)}\sqrt{1+L_T}}\left(-\frac{\partial B_r^{(0)}}{\partial z}\right) \tag{2.8}$$

$$\left(\frac{\partial}{\partial r} + ik_T^{(0)} - ik_T^{(0)}\sqrt{1+L_T}\right) B_z^{(0)} = \frac{1}{2ik_T^{(0)}\sqrt{1+L_T}}\left(-\frac{\partial B_\theta^{(0)}}{\partial \theta}\right). \tag{2.9}$$

In the above equations, the operators L_L and L_T are defined by

$$L_L = \frac{1}{(k_L^{(0)})^2}\left(\frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \tag{2.10}$$

$$L_T = \frac{1}{(k_T^{(0)})^2}\left(\frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right). \tag{2.11}$$

The longitudinal wave number $k_L^{(0)}$ and the transverse wave number $k_T^{(0)}$ are given by

$$(k_L^{(0)})^2 = \frac{\rho^{(0)}\omega^2}{\lambda^{(0)} + 2\mu^{(0)}} \tag{2.12}$$

$$(k_T^{(0)})^2 = \frac{\rho^{(0)}\omega^2}{\mu^{(0)}}, \tag{2.13}$$

where ω is radian frequency, $\rho^{(0)}$ is the mean (average) density and $\lambda^{(0)}$ and $\mu^{(0)}$ are the mean Lamé parameters. The first-order equations which account for the perturbations in density and in the Lamé parameters have a form very similar to Eqs. (2.6)-(2.9), and thus the numerical techniques discussed for the zeroth-order equations in this paper can be applied to the first-order equations as well. To simplify the notation, the superscript zero on the longitudinal and transverse wave numbers and on the envelope functions A, B_r, B_θ and B_z will subsequently be omitted.

3 An ODE Formulation

We formulate Eqs. (2.6) through (2.9) into a system of equations in an ordinary differential equation form. We define

$$\alpha_L = P_L + Q_L \tag{3.1}$$

$$\alpha_T = P_T + Q_T \tag{3.2}$$

$$\beta_T = R_T + S_T, \tag{3.3}$$

where

$$P_L = \frac{1}{2k_L}\frac{\partial^2}{\partial z^2} - \frac{1}{8k_L^3}\frac{\partial^4}{\partial z^4} \tag{3.4}$$

$$P_T = \frac{1}{2k_T}\frac{\partial^2}{\partial z^2} - \frac{1}{8k_T^3}\frac{\partial^4}{\partial z^4} \tag{3.5}$$

$$Q_L = \frac{1}{2k_L}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{3.6}$$

$$Q_T = \frac{1}{2k_T}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{3.7}$$

$$R_T = \frac{1}{2k_T} - \frac{1}{4k_T^3}\frac{\partial^2}{\partial z^2} + \frac{3}{16k_T^5}\frac{\partial^4}{\partial z^4} \tag{3.8}$$

$$S_T = -\frac{1}{4k_T^3}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{3.9}$$

The operators α_L and α_T are wide-angle approximations [4] to the operators involving $\sqrt{1+L_L}$ and $\sqrt{1+L_T}$, respectively, and the operator β_T is the corresponding wide-angle approximation to the operator involving $1/\sqrt{1+L_T}$. With the above approximations and definitions, Eqs. (2.6)-(2.9) can be written in matrix form as

$$\frac{\partial}{\partial r}\begin{pmatrix} A \\ B_r \\ B_\theta \\ B_z \end{pmatrix} = i \begin{pmatrix} \alpha_L & 0 & 0 & 0 \\ 0 & \alpha_T & 0 & 0 \\ 0 & \beta_T \frac{\partial}{\partial z} & \alpha_T & 0 \\ 0 & 0 & \beta_T \frac{\partial}{\partial \theta} & \alpha_T \end{pmatrix} \begin{pmatrix} A \\ B_r \\ B_\theta \\ B_z \end{pmatrix}. \tag{3.10}$$

Equation (3.10) is in the form of a first-order system of ordinary differential equations. In the following section we present an efficient method for solving Eq. (3.10) numerically.

4 A Numerical ODE Solution

We look not only for a solution of Eq. (3.10), but also for an efficient solution; that is, a solution which maintains the required accuracy while at the same time minimizes memory requirements and computation time. It is possible to consider numerical methods which solve Eq. (3.10) as a system, but such methods require large memory capacity and a correspondingly slow computation time. It is much simpler to note that the lower triangular form of the 4×4 matrix in Eq. (3.10) allows a method which solves each individual scalar equation sequentially. By developing a sequential scalar approach, we may take advantage of the efficient and well-developed techniques which have been derived for solving wave equations in scalar acoustics. More importantly, the coupling in the matrix system is explicitly maintained.

Each scalar equation in Eq. (3.10) can be written in the form

$$\begin{aligned}\frac{\partial}{\partial r}f &= i\alpha f + i\beta g \\ &= i(P+Q)f + i(R+S)g, \end{aligned} \tag{4.1}$$

where the general operators α, β, P, Q, R and S can represent either the logitudinal operators (subscript L) or the transverse operators (subscript T) defined in Sec. 3. The forcing term g is assumed to be known. For the A and B_r equations in Eq. (3.10), the forcing term g is zero. If the A and B_r equations in are solved first, then the forcing term g in the B_θ equation depends on the known function B_r. After the B_θ equation has been solved, the forcing term g in the B_z equation depends on the known function B_θ. This is where the coupling is maintained. We will give a complete solution method for the specific equation

$$\begin{aligned}\frac{\partial}{\partial r}B_\theta &= i\alpha_T B_\theta + i\beta_T\frac{\partial B_r}{\partial z} \\ &= i(P_T+Q_T)B_\theta + i(R_T+S_T)\frac{\partial B_r}{\partial z}. \end{aligned} \tag{4.2}$$

This solution method can be applied to each of the four scalar equations in Eq. (3.10).

4.1 The NLMS (NonLinear MultiStep) Method

In the NLMS (NonLinear MultiStep) method, a discretized solution of Eq. (4.2) is written in the form

$$\begin{aligned}(B_\theta)^{n+1} &= e^{i\alpha_T(\Delta r)}(B_\theta)^n + (\Delta r)i\beta_T\left(\frac{\partial B_r}{\partial z}\right)^n \\ &= e^{i(P_T+Q_T)(\Delta r)}(B_\theta)^n + (\Delta r)i\beta_T\left(\frac{\partial B_r}{\partial z}\right)^n \\ &= e^{iP_T(\Delta r)}e^{iQ_T(\Delta r)}(B_\theta)^n + (\Delta r)i\beta_T\left(\frac{\partial B_r}{\partial z}\right)^n, \end{aligned} \tag{4.3}$$

where the superscript refers to the discrete index corresponding to the range coordinate r, Δr is the spatial increment in the range direction, and where the commutativity between $e^{iP_T(\Delta r)}$ and $e^{iQ_T(\Delta r)}$ is assumed. This is a simplified first-order explicit scheme, which has been successfully applied to two- and three-dimensional acoustic equations in refs. [2] and [1]. The theory of stability and convergence for this scheme has been well established, as discussed in ref. [1].

In order to solve Eq. (4.3), it is necessary to develop expressions for the exponential operators $e^{iP_T(\Delta r)}$ and $e^{iQ_T(\Delta r)}$. Using Eq. (3.4), the operator $e^{iP_T(\Delta r)}$ can be written as

$$e^{iP_T(\Delta r)} = e^{\delta_r(\frac{1}{2}\Gamma_z - \frac{1}{8}\Gamma_z^2)}, \tag{4.4}$$

where

$$\delta_r = ik_T(\Delta r) \tag{4.5}$$

and

$$\Gamma_z = \frac{1}{k_T^2}\frac{\partial^2}{\partial z^2}. \tag{4.6}$$

We now take advantage of using the rational function approximation [4]

$$e^{\delta_r(\frac{1}{2}\Gamma_z - \frac{1}{8}\Gamma_z^2)} = \frac{1 + (\frac{1}{4} + \frac{\delta_r}{4})\Gamma_z}{1 + (\frac{1}{4} - \frac{\delta_r}{4})\Gamma_z} + O(\Gamma_z^3). \tag{4.7}$$

It will be seen below that this rational function approximation leads to a highly efficient finite difference scheme.

Using Eq. (3.7), the operator $e^{iQ_T(\Delta r)}$ in Eq. (4.3) can be written as

$$e^{iQ_T(\Delta r)} = e^{\delta_r \frac{1}{2}\Gamma_\theta}, \tag{4.8}$$

where

$$\Gamma_\theta = \frac{1}{k_T^2}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{4.9}$$

Here we use the Padé (1,1) rational function approximation to obtain

$$e^{\delta \frac{1}{2}\Gamma_\theta} = \frac{1 + \frac{\delta_r}{4}\Gamma_\theta}{1 - \frac{\delta_r}{4}\Gamma_\theta} + O(\Gamma_\theta^2). \tag{4.10}$$

With the preceding rational function approximations for $e^{iP_T(\Delta r)}$ and $e^{iQ_T(\Delta r)}$, Eq. (4.3) becomes

$$(B_\theta)^{n+1} = \left(\frac{1 + (\frac{1}{4} + \frac{\delta_r}{4})\Gamma_z}{1 + (\frac{1}{4} - \frac{\delta_r}{4})\Gamma_z}\right)\left(\frac{1 + \frac{\delta_r}{4}\Gamma_\theta}{1 - \frac{\delta_r}{4}\Gamma_\theta}\right)(B_\theta)^n + (\Delta r)i\beta_T\left(\frac{\partial B_r}{\partial z}\right)^n \tag{4.11}$$

or

$$\left[1 + \left(\frac{1}{4} - \frac{\delta_r}{4}\right)\Gamma_z\right]\left[1 - \frac{\delta_r}{4}\Gamma_\theta\right](B_\theta)^{n+1} =$$
$$\left[1 + \left(\frac{1}{4} + \frac{\delta_r}{4}\right)\Gamma_z\right]\left[1 + \frac{\delta_r}{4}\Gamma_\theta\right](B_\theta)^n + (\Delta r)i\beta_T\left[1 + \left(\frac{1}{4} - \frac{\delta_r}{4}\right)\Gamma_z\right]\left[1 - \frac{\delta_r}{4}\Gamma_\theta\right]\left(\frac{\partial B_r}{\partial z}\right)^n. \tag{4.12}$$

We rewrite Eq. (4.12) as

$$\left[1 + \left(\frac{1}{4} - \frac{\delta_r}{4}\right)\Gamma_z\right](V)^{n+1} = (g)^n = (g_1)^n + (g_2)^n, \tag{4.13}$$

where

$$(V)^{n+1} = \left(1 - \frac{\delta_r}{4}\Gamma_\theta\right)(B_\theta)^{n+1} \tag{4.14}$$

$$(g_1)^n = \left[1 + \left(\frac{1}{4} + \frac{\delta_r}{4}\right)\Gamma_z\right]\left[1 + \frac{\delta_r}{4}\Gamma_\theta\right](B_\theta)^n \tag{4.15}$$

$$(g_2)^n = \left[1 + \left(\frac{1}{4} - \frac{\delta_r}{4}\right)\Gamma_z\right]\left[1 - \frac{\delta_r}{4}\Gamma_\theta\right] i(\Delta r)\beta_T \left(\frac{\partial B_r}{\partial z}\right)^n . \tag{4.16}$$

The finite difference scheme for solving Eq. (4.12) consists of two steps. In the first step, central difference approximations are applied to the differential operators in Eq. (4.13), and Eq. (4.13) is solved for $(V)^{n+1}$. In the second step, central difference approximations are applied to the differential operators in Eq. (4.14), and Eq. (4.14) is solved for $(B_\theta)^{n+1}$. Step one involves the z-derivatives contained in the operator Γ_z, and step two involves the θ-derivatives contained in the operator Γ_θ. It will be shown that in each step it is only necessary to solve a tri-diagonal system of linear algebraic equations.

We now give the finite difference equations needed to implement the solution of Eq. (4.13) and Eq. (4.14). In the equations which follow, we continue to use a superscript to refer to the discrete index of the range coordinate r. The first subscript refers to the index of the depth coordinate z, and the second subscript refers to the discrete index of the azimuthal coordinate θ. The quantity $\Delta\theta$ is the increment in the θ-coordinate, and the quantity Δz is the increment in the z-coordinate. We also define

$$\delta_z = k_T(\Delta z) \tag{4.17}$$

$$\delta_\theta = k_T r(\Delta\theta). \tag{4.18}$$

The expression for the term g_1 then becomes

$$\begin{aligned}
(g_1)^n_{m,l} &= \left[1 + \left(\frac{1}{4} + \frac{\delta_r}{4}\right)\frac{1}{k_T^2}\frac{\partial^2}{\partial z^2}\right]\left[1 + \frac{\delta_r}{4}\frac{1}{k_T^2}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right](B_\theta)^n_{m,l} \\
&= \left[1 + \left(\frac{1}{4} + \frac{\delta_r}{4}\right)\frac{1}{k_T^2}\frac{\partial^2}{\partial z^2}\right] \\
&\qquad \cdot \left[\frac{\delta_r}{4\delta_\theta^2}(B_\theta)^n_{m,l+1} + \left(1 - \frac{\delta_r}{2\delta_\theta^2}\right)(B_\theta)^n_{m,l} + \frac{\delta_r}{4\delta_\theta^2}(B_\theta)^n_{m,l-1}\right] \\
&= \left[\frac{1}{4\delta_z^2} + \frac{\delta_r}{4\delta_z^2}\right] \\
&\qquad \cdot \left[\frac{\delta_r}{4\delta_\theta^2}(B_\theta)^n_{m+1,l+1} + \left(1 - \frac{\delta_r}{2\delta_\theta^2}\right)(B_\theta)^n_{m+1,l} + \frac{\delta_r}{4\delta_\theta^2}(B_\theta)^n_{m+1,l-1}\right] \\
&\quad + \left[1 - \frac{1}{2\delta_z^2} - \frac{\delta_r}{2\delta_z^2}\right]
\end{aligned}$$

$$\cdot \left[\frac{\delta_r}{4\delta_\theta^2}(B_\theta)^n_{m,l+1} + \left(1 - \frac{\delta_r}{2\delta_\theta^2}\right)(B_\theta)^n_{m,l} + \frac{\delta_r}{4\delta_\theta^2}(B_\theta)^n_{m,l-1}\right]$$
$$+ \left[\frac{1}{4\delta_z^2} + \frac{\delta_r}{4\delta_z^2}\right]$$
$$\cdot \left[\frac{\delta_r}{4\delta_\theta^2}(B_\theta)^n_{m-1,l+1} + \left(1 - \frac{\delta_r}{2\delta_\theta^2}\right)(B_\theta)^n_{m-1,l} + \frac{\delta r}{4\delta_\theta^2}(B_\theta)^n_{m-1,l-1}\right]. \quad (4.19)$$

Next, we consider the term g_2:

$$\begin{aligned}
(g_2)^n &= i(\Delta r)\left[1 + \left(\frac{1}{4} - \frac{\delta_r}{4}\right)\frac{1}{k_T^2}\frac{\partial^2}{\partial z^2}\right]\left[1 - \frac{\delta_r}{4}\frac{1}{k_T^2}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\beta_T\left(\frac{\partial B_r}{\partial z}\right)^n \\
&= i(\Delta r)\left[1 + \left(\frac{1}{4} - \frac{\delta_r}{4}\right)\frac{1}{k_T^2}\frac{\partial^2}{\partial z^2}\right]\left[1 - \frac{\delta_r}{4}\frac{1}{k_T^2}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] \\
&\quad \cdot \left[\frac{1}{2k_T} - \frac{1}{4k_T^3}\frac{\partial^2}{\partial z^2} + \frac{3}{16k_T^5}\frac{\partial^4}{\partial z^4} - \frac{1}{4k_T^3}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\left(\frac{\partial B_r}{\partial z}\right)^n \\
&= i(\Delta r)\left[1 + \left(\frac{1}{4} - \frac{\delta_r}{4}\right)\frac{1}{k_T^2}\frac{\partial^2}{\partial z^2}\right]\left[1 - \frac{\delta_r}{4}\frac{1}{k_T^2}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] \\
&\quad \cdot \left[\frac{1}{2} - \frac{1}{4k_T^2}\frac{\partial^2}{\partial z^2} + \frac{3}{16k_T^4}\frac{\partial^4}{\partial z^4} - \frac{1}{4k_T^2}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\frac{1}{k_T}\left(\frac{\partial B_r}{\partial z}\right)^n.
\end{aligned} \quad (4.20)$$

The explicit finite difference expression for g_2 is straightforward to derive but is very long, and we will not give the expression here. Once the expression for $g = g_1 + g_2$ has been obtained, Eq. (4.13) is

$$\left[1 + \left(\frac{1}{4} - \frac{\delta_r}{4}\right)\frac{1}{k_T^2}\frac{\partial^2}{\partial z^2}\right](V)^{n+1}_{l,m} = (g)^n_{l,m}, \quad (4.21)$$

where the right-hand side is now known. Using a central difference approximation for the z-derivative, Eq. (4.21) becomes

$$\left(\frac{1}{4} - \frac{\delta_r}{4}\right)\frac{1}{\delta_z^2}(V)^{n+1}_{l-1,m} + \left[1 + \left(\frac{1}{4} - \frac{\delta_r}{4}\right)\frac{2}{\delta_z^2}\right](V)^{n+1}_{l,m} + \left(\frac{1}{4} - \frac{\delta_r}{4}\right)\frac{1}{\delta_z^2}(V)^{n+1}_{l+1,m} = (g)^n_{l,m}. \quad (4.22)$$

For each value of the range index n and the azimuthal index m, Eq. (4.22) is a tridiagonal system of equations in the depth index l. Efficient methods for solving such a sytem are discussed in ref. [4].

When the values of $(V)^{n+1}$ have been computed from Eq. (4.22), Eq. (4.14) becomes

$$\left[1 + \left(\frac{1}{4} - \frac{\delta_r}{4}\right)\frac{1}{r^2}\frac{1}{k_T^2}\frac{\partial^2}{\partial \theta^2}\right](B_\theta)^{n+1}_{l,m} = (V)^{n+1}_{l,m}, \quad (4.23)$$

where the right-hand is now known. Using a central difference approximation for the θ-derivative, Eq. (4.23) becomes

$$-\frac{\delta_r}{4}\frac{1}{\delta_\theta^2}(B_\theta)_{l,m-1}^{n+1}+\left(1+\frac{\delta_r}{4}\frac{2}{\delta_\theta^2}\right)(B_\theta)_{l,m}^{n+1}-\frac{\delta_r}{4}\frac{1}{\delta_\theta^2}(B_\theta)_{l,m+1}^{n+1}=(V)_{l,m}^{n+1}. \quad (4.24)$$

For each value of the range index $n+1$ and the depth index l, Eq. (4.24) is a tridiagonal system of equations in the azimuthal index m. Equation (4.24) may be solved with the same methods used to solve Eq. (4.22).

In summary, the finite difference solution of Eq. (4.2) is computed as follows. First, the expressions for g_1 and g_2 are computed using Eqs. (4.19) and (4.20). Second, the tridiagonal system represented by Eq. (4.22) is solved for $(V)^{n+1}$. Finally, when $(V)^{n+1}$ has been obtained, the tridiagonal system represented by Eq. (4.24) is solved for $(B)_\theta^{n+1}$. The entire process is repeated for each range increment and for each of the four scalar equations in Eq. (3.10), thus giving a marching procedure for computing the complete elastic field.

4.2 Remarks on the Selection of Range Stepsize

The numerical scheme based on Eq. (4.3), which we apply to solve the ordinary differential equation given in Eq. (4.2), is a 1st-order nonlinear mustistep as described in ref. [1]. For the application of Eq. (4.3) to solve Eq. (4.2), the range step size must be selected such that is satisfies the stability condition

$$\Delta r < \frac{1}{L_g}, \quad (4.25)$$

where the L_g is the Lipschitz constant for $\| \frac{\partial g}{\partial u} \|$ [1]. Note that in the case when the g function is zero, as in the first two scalar equations of Eq. (3.10), any step size can be selected; this means the scheme based on Eq. (4.3) is unconditionally stable, since the rational function operator is unitary. If g is not zero, the scheme based on Eq. (4.3) is conditionally stable.

5 CONCLUSIONS

The set of elastic wave equation given in Eqs. (2.6)-(2.9) was developed to be useful for numerical calculations and to be consistent with the formulation of three-dimensional fluid wave equations. This makes the elastic equations adaptable for the development a complete model for the study of fluid-elastic interface problems. However, it is necessary to solve the elastic equations accurately and efficiently. The numerical schemes introduced here have been very well developed in theory and in the practical solutions of purely fluid problems. The fact that the elastic equations can be formulated and solved with only small modifications makes the scheme we have developed here very desirable for problems involving both fluid and elastic wave equations.

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