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Advances in Space-Time Finite Element Methods for Structural Acoustics and Fluid-Solid Interaction

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Abstract

Traditional computational approaches toward simulation of radiation and scattering from elastic bodies submerged in an acoustic fluid have been primarily based on frequency domain formulations. Classical time-harmonic approaches (including boundary element, finite element, and finite difference methods) have been effective for problems involving a limited number of frequencies (narrow band response) and scales (wavelengths) that are large compared to the characteristic dimensions of the elastic structure. Attempts at solving large-scale structural acoustic systems with dimensions that are much larger than the operating wavelengths and which are complex, consisting of many different components with different scales and broadband frequencies, has revealed limitations of many of the classical methods. As a result, in recent times there has been renewed interest in new and alternative approaches, including time-domain approaches. This paper describes recent advances in the development of a new class of high-order accurate and unconditionally stable space-time methods which employ finite element discretization of the time domain as well as the usual discretization of the spatial domain. The formulation is based on a space-time variational equation for both the acoustic fluid and elastic solid together with their interaction. This novel approach to the modeling of the temporal variables allows for the consistent use of high-order adaptive solution strategies for unstructured grids in both time and space; a technology that is vital for the efficiency of the resulting computational algorithm. Another important feature is the incorporation of temporal jump operators which allow for discretizations that are discontinuous in time. The specific form of these jump operators are designed to capture multiple scales in the response of large-scale structural acoustic systems. For additional stability, least-squares operators based on local residuals of the Euler-Lagrange equations including non-reflecting boundary conditions are incorporated. Topics to be discussed include the development and implementation of new higher-order accurate non-reflecting boundary conditions based on the exact impedance relation through the Dirichlet-to-Neumann (DtN) map, multi-field representations based on acoustic pressure and velocity potential variables, error estimation and stability.

Introduction

Most industrial calculations of transient structural dynamic and wave propagation problems in the time-domain have used boundary element methods based on Kirchhoff's retarded potential integral formulation, and varients of traditional finite element schemes employing standard Galerkin methods in space and finite difference techniques for integrating in time, that feature low-order accurate solvers, and low-order approximate non-reflecting boundaries. In recent years, dramatically different approaches to these types of numerical simulations have been advanced [1, 2, 3, 4, 5, 6, 7, 8, 9] which use emerging finite element technologies. The principal features of these space-time methods are that they employ: (i) higher-order approximations in both space and time dimensions, (ii) unstructured meshes

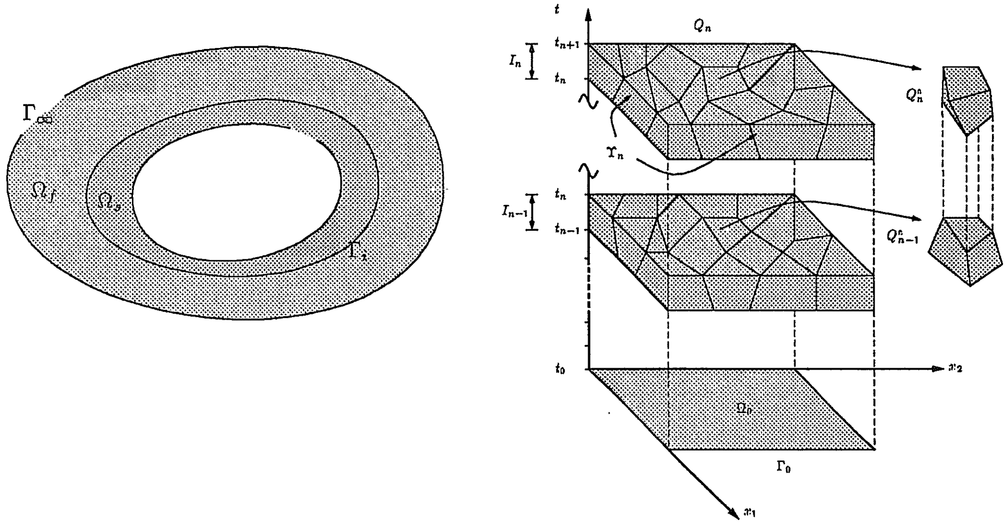

Figure 1: (Left): Coupled system for the exterior fluid-structure interaction problem, with artificial boundary Γ_∞ enclosing the finite computational domain $\Omega = \Omega_f \cup \Omega_s$. (Right): Illustration of two consecutive space-time slabs with unstructured finite element meshes within a slab.

in both space and time, (iv) physically based dissipative mechanisms, (v) higher-order non-reflecting boundary conditions, and (vi) iterative solvers for high-performance parallel computation. Collectively, these features give promise of significant advances in efficiency, reliability, and flexibility in simulation software designed for transient wave-propagation. High-order space-time finite element methods are capable of delivering very high accuracies for wave propagation simulations, particularly for problems involving sharp gradients in the solution which typically arise in the vicinity of fluid-structure interfaces and near inhomogeneities such as stiffeners, structural joints, and material discontinuities. In these problems, solutions obtained with standard numerical methods may have difficulty resolving the discontinuities in the physical solution – in the case of standard time integrators, large spurious oscillations may appear which pollute the entire solution. In addition, for problems involving the propagation of pulses with broadband frequencies over a large distance, commonly used second-order accurate numerical algorithms may exhibit significant dispersion errors causing misrepresentation of arrival time and directionality at a distant target. In this paper, recent developments in the application high-order accurate space-time finite element methods for structural acoustics are presented.

The Structural Acoustics Problem

Consider the coupled system consisting of a structural region Ω_s surrounded by an infinite fluid region $\mathcal{B}$. The interface boundary between the structure and fluid domains is denoted by Γ_i. The unit outward normal to the structure (inward normal to the fluid) on Γ_i is denoted by n. The non-reflecting boundary is denoted Γ_∞ and positioned such that the original fluid region $\mathcal{B}$ is divided into a bounded interior domain Ω_f and an exterior domain Ω_∞ such that $\mathcal{B} = \Omega_f \cup \Omega_\infty$; see Fig. 1. The structure is assumed to be governed by the equations of elastodynamics while the fluid equations are taken under the usual linear acoustic assumptions of an inviscid, compressible fluid with small disturbance. The momentum equations for the fluid are

$$\nabla p + \rho_f \dot{v} = 0 \tag{1}$$

where $p(x\,,\,t)$ is the acoustic pressure, $v(x\,,\,t)$ is the fluid particle velocity, and $\rho_f(x) > 0$ is the density of the fluid. A superimposed dot indicates partial differentiation with respect to time t. The constitutive behavior of the fluid is assumed to be

$$\dot{p} + K_f \nabla \cdot v = 0 \tag{2}$$

where $K_f = \rho_f c^2$ is the bulk modulus and c is the acoustic wave speed. From the assumption of an irrotational acoustic fluid, the velocity can be written as the gradient of the velocity potential ϕ as $v = \nabla\phi$. Consequently, pressure is related to the velocity potential by $p = -\rho_f \dot{\phi}$. On the structural interface Γ_i, the normal component of the fluid velocity is assumed to be equivalent to the motion of the structural surface. Projecting the velocity normal to the structure gives the fluid-structure coupling: $v \cdot n = v_s \cdot n$ where $v_s(x\,,\,t)$ is the structural velocity vector. The influence of the fluid pressure acting on the structure is given by the normal traction $\sigma \cdot n = -pn$ where σ is the symmetric Cauchy stress tensor. The stress is assumed to be related to the structural displacement vector $u_s(x\,,\,t)$ through a linear constitutive relation of the form:

$$\sigma(u_s) = C : \nabla^s u_s \tag{3}$$

where $\nabla^s u_s$ is the symmetric gradient and $C = C(x)$ is the fourth-order tensor of elastic coefficients; assumed to satisfy the usual pointwise stability and major and minor symmetry properties. The equations of motion for the structure are

$$\nabla \cdot \sigma = \rho_s \dot{v}_s \tag{4}$$

where $\rho_s(x) > 0$ is the structural density, and $v_s(x\,,\,t) = \dot{u}_s(x\,,\,t)$. The drivers for the problem are the initial conditions:

$$u_s(x,0) = u_s^0(x) \;; \qquad v_s(x,0) = v_s^0(x) \qquad x \in \Omega_s \tag{5}$$

$$\phi(x,0) = \phi^0(x) \;; \qquad p(x,0) = p^0(x) \qquad x \in \Omega_f \tag{6}$$

Exact Non-Reflecting Boundary Conditions

In this paper an exact non-reflecting boundary condition is used as a basis for the space-time finite element formulation. An exact non-reflecting boundary condition is obtained by taking advantage of the fact that an outgoing wave solution can always be written in terms of a series of wave harmonics with respect to a separable coordinate system. In the frequency domain, i.e., the time-harmonic problem, this idea has been exploited by several researchers to derive exact non-reflecting boundary conditions; see e.g. the Dirichlet-to-Neumann (DtN) impedance operator derived in [10]. The DtN operator is a nonlocal (integral) and frequency dependent boundary condition applied on a separable boundary Γ_∞.

For a spherical boundary Γ_∞ of radius $r = R$ with unit outward normal n to Γ_∞, the exact representation of the exterior acoustic impedance restricted to Γ_∞ is,

$$v(R,\theta,\varphi)\cdot n = \sum_{n=0}^{N-1} z_n(\hat{k}) \int_{\Gamma_\infty} s_n(\theta,\varphi,\theta',\varphi')\, \phi(R,\theta',\varphi')\, d\Gamma' \tag{7}$$

where the DtN kernels s_n, $n = 0, 1, 2, \cdots$ are given by,

$$s_n = \sum_{j=0}^{n}{}' \frac{(2n+1)(n-j)!}{2\pi R^2 (n+j)!} P_n^j(\cos\varphi) P_n^j(\cos\varphi') \cos j(\theta - \theta') \tag{8}$$

with impedance coefficients,

$$z_n(\hat{k}) = \frac{k h_n'(\hat{k})}{h_n(\hat{k})} \tag{9}$$

In the above, $\omega > 0$ is the frequency, $k = \omega/c$ is the acoustic wavenumber, $\hat{k} = kR$, $0 \le \theta < 2\pi$ is the circumferential angle and $0 \le \varphi < \pi$ is the polar angle for a spherical truncation boundary. The differential surface area is $d\Gamma = J_s d\theta d\varphi$, where $J_s = R^2 \sin\varphi$. The functions P_n^j are associated Legendre functions of the first kind, and h_n are spherical Hankel functions of the first kind of order n. The prime on h_n indicates differentiation with respect to its argument, and the prime after the sum indicates that a factor of 1/2 multiplies the term with $j = 0$. The boundary condition (7) is exact when $N = \infty$. A direct time-dependent counterpart to (7) can be obtained through a convolution integral in time resulting in a boundary condition that is non-local in both space and time dimensions. Implementation in a computational method requires storage of all previous solutions up to the current time step; a property that makes its use impractical for large-scale computations over long time intervals. Note that this limitation of the time convoluted DtN operator is also shared with the Kirchoff boundary integral representation.

In order to circumvent the difficulty of having to implement a temporal convolution integral, time-dependent boundary conditions have been derived which replace the temporal integral with local temporal derivatives; [5, 7]. Two alternative sequences were derived; the first retains the nonlocal spatial integral of the DtN map (7), while replacing the time-convolution with higher-order local time derivatives (local in time and nonlocal in space version), while the second involves only time and spatial derivatives (local in time and local in space version).

Local in Time and Non-Local in Space Version

In [5, 7] it is shown that by taking advantage of recurrence formulus for the Hankel functions appearing in the impedance coefficients (9), an alternative form of the truncated DtN map (7) can be obtained which has an exact inverse Fourier transform. Using this procedure, a family of high-order accurate and time-dependent non-reflecting boundary conditions are obtained which share the property of the DtN map, i.e. the boundary conditions match the first N spherical harmonics for outgoing waves on a spherical boundary Γ_∞. For example, for $N = 2$ the time-dependent counterpart to (7) is,

$$\begin{aligned} K_f \boldsymbol{v} \cdot \boldsymbol{n} &= c^2 p_{,n} + \frac{1}{R} \int_{\Gamma_\infty} \left(R^2 p_{,t} + 2cRp - K_f \phi \right) s_0 \, d\Gamma' \\ &+ \frac{1}{R} \int_{\Gamma_\infty} \left(R^2 p_{,t} + 2cRp - 2K_f \phi \right) s_1 \, d\Gamma' \end{aligned} \tag{10}$$

This time-dependent radiation boundary condition is perfectly absorbing for the first two spherical wave harmonics of order $n = 0$ and $n = 1$. Formulas for higher-order operators are reported in [5, 7]. These boundary conditions retain the nonlocal spatial integral, yet replace the time-convoluted DtN map with high-order local time derivatives. This form of time-dependent boundary condition has the advantage that when implemented in the time-discontinuous finite element formulation, standard $C^0(\Gamma_\infty \times I_n)$ basis functions may be used for both the space and time variables.

Local in Time and Local in Space Version

An alternate version is obtained by first localizing the acoustic impedance relation (7) in the frequency domain, followed by an inverse Fourier transform. In [5, 7] it is shown that when the solution on the boundary Γ_∞ contains only a finite number of spherical harmonics, then such a transformation gives an exact time-dependent counterpart which is local in both space x and time t. The transformation starts with the ideas of Givoli and Keller [11], where a spatially local counterpart to the non-local DtN map was obtained in two-dimensions. The extension to three-dimensions was given by Harari [12]. The development proceeds by recognizing that the spherical harmonics can be interpreted as eigenfunctions of the Laplace-Beltrami operator

$$\Delta_\Gamma := \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2} \tag{11}$$

Using this result, the DtN map (7) can be written in the following localized form:

$$v \cdot n = \sum_{m=0}^{N-1} \beta_m(\hat{k})\,(-\Delta_\Gamma)^m \phi \qquad \text{on } \Gamma_\infty \tag{12}$$

where the values of $\beta_m(\hat{k})$ are obtained by solving the $N \times N$ linear algebraic system,

$$z_n(\hat{k}) = \sum_{m=0}^{N-1} [n(n+1)]^m \beta_m(\hat{k}), \qquad n = 0, 1, \cdots, N-1 \tag{13}$$

Since this sequence follows directly from the truncated DtN map, these radiation boundary operators annilate the first N spherical harmonics for the outgoing solution on a spherical boundary Γ_∞.

Local time-dependent counterparts to (12) have been obtained in [5, 7] using an inverse Fourier transform procedure by first solving (13) for the coefficients β_m in terms of the impedance coefficients z_n and then using recurrence relations for the spherical Hankel functions h_n to simplify the result. For example, for $N = 2$, (13) can be solved for $\beta_0 = z_0$ and $\beta_1 = (z_1 - z_0)/2$, leading to the result:

$$K_f\, v \cdot n = cRp_{,n} + Rp_{,t} + 2cp - \frac{K_f}{2R}(2 - \Delta_\Gamma)\phi \tag{14}$$

When applied on a spherical boundary Γ_∞, this operator acts as a high-order accurate local boundary condition which is perfectly absorbing for the first two spherical wave harmonics of orders $n = 0$ and $n = 1$. As the order N is increased, i.e. more terms are used in (12), the resulting boundary conditions match more terms in the harmonic expansion for outgoing waves, and a better approximation is obtained: see [5, 7] for expressions for the time-dependent counterparts to (12) for $N \geq 3$. These boundary conditions are implemented in the multi-field space-time finite element formulation as natural boundary conditions, i.e., they are enforced weakly in both time and space. We note that the operator defined in (14) is identical to the second-order radiation boundary condition derived by Bayliss and

Turkel in [13], after second-order radial derivatives are eliminated in favor of second-order tangential derivatives through use of the wave equation in spherical coordinates. Thus, while the boundary conditions derived by Bayliss and Turkel were obtained by annilating radial terms in a multipole expansion, it is seen, that in fact, the first two boundary conditions in the sequence share the property of the localized DtN, in that they match the first two spherical harmonics for outgoing waves on a spherical boundary Γ_∞. For higher-order boundary conditions in the sequence beyond $N \geq 3$, the form of the boundary conditions derived in [5, 7] differ from those derived in [13]. Because the time-discontinuous formulation allows for the use of $C^0(I_n)$ interpolations to represent the high-order time derivatives, it is possible to implement these sequences of time-dependent absorbing boundary conditions up to any order desired. However for higher-order operators extending beyond $N \geq 3$, the lowest possible order of spatial continuity on the artificial boundary that can be achieved after integration by parts is C^{N-2}. For these high-order operators a layer of boundary elements adjacent to Γ_∞, possessing high-order tangential continuity on Γ_∞ are needed, see e.g. [14].

Time-Discontinuous Galerkin FE Formulation

The development of the space-time method proceeds by considering a partition of the time interval, $I =]0, T[$, of the form: $0 = t_0 < t_1 < \ldots < t_N = T$, with $I_n =]t_n, t_{n+1}[$ and $\Delta t_n = t_{n+1} - t_n$. Using this notation, $Q_n^s = \Omega_s \times I_n$, and $Q_n^f = \Omega_f \times I_n$ are the nth space-time slabs for the structure and fluid respectively. For the nth space-time slab, the spatial domain is subdivided into $(n_{el})_n$ elements, and the interior of the e^{th} element is defined as Q_n^e. Figure 1 shows an illustration of two consecutive space-time slabs Q_{n-1} and Q_n. Within each space-time element, the trial solution and weighting function are approximated by polynomials in both x and t. These functions are assumed $C^0(Q_n)$ continuous throughout each space-time slab, but are allowed to be discontinuous across the interfaces of the slabs. The space of finite element basis functions for the multi-field representation for the fluid are stated in terms of *independent* trial velocity potential ϕ^h, and trial pressure p^h, variables:

Trial velocity potential:

$$\mathcal{T}_1^h = \left\{ \phi^h \middle| \phi^h \in C^0\Big(\bigcup_{n=0}^{N-1} Q_n^f\Big), \phi^h\big|_{Q_n^{f^e}} \in \mathcal{P}^k(Q_n^{f^e}) \right\}$$

Trial pressure:

$$\mathcal{T}_2^h = \left\{ p^h \middle| p^h \in C^0\Big(\bigcup_{n=0}^{N-1} Q_n^f\Big), p^h\big|_{Q_n^{f^e}} \in \mathcal{P}^l(Q_n^{f^e}) \right\}$$

where $\mathcal{P}^k$ denotes the space of kth-order polynomials and C^0 denotes the space of continuous functions. Similar collections of finite element basis functions for the approximation of independent structural displacements u_s^h and velocities v_s^h are defined. An important component in the success of the space-time method is the incorporation of discontinuous temporal jump terms at each space-time slab interface; for a function w^h, the jump operator is defined as,

$$[\![w^h(t_n)]\!] = w^h(x, t_n^+) - w^h(x, t_n^-)$$

These jump operators weakly enforce initial conditions across time slabs and are crucial for obtaining an unconditionally stable algorithm for unstructured space-time finite element discretizations using high-order interpolations.

In the following, we shall use the following notation:

$$\begin{aligned}
(\delta u_s^h\,,\,u_s^h)_{\Omega_s} &= \int_{\Omega_s} \delta u_s^h \cdot u_s^h \, d\Omega \\
a(\delta u_s^h\,,\,u_s^h)_{\Omega_s} &= \int_{\Omega_s} \nabla \delta u_s^h \cdot \sigma(u_s^h) \, d\Omega \\
(\delta p^h\,,\,p^h)_{\Omega_f} &= \int_{\Omega_f} \delta p^h \, p^h \, d\Omega \\
(\delta p^h\,,\,p^h)_{Q_n} &= \int_{t_n}^{t_{n+1}} (\delta p^h\,,\,p^h)_{\Omega} \, dt \\
(\delta p^h\,,\,p^h)_{\Upsilon_n} &= \int_{t_n}^{t_{n+1}} (\delta p^h\,,\,p^h)_{\Gamma} \, dt
\end{aligned}$$

A delta refers to the variation of the function, i.e. the corresponding weighting function, and the L_2 norm is denoted $||\phi||_\Omega = (\phi\,,\,\phi)_\Omega^{1/2}$.

A Multi-field space-time variational equation

A multi-field space-time variational equation for the exterior structural acoustics problem is obtained from a weighted residual of the governing equations and incorporates time-discontinuous jump terms. For efficiency the method is applied in one space-time slab at a time; data from the end of the previous slab are employed as initial conditions for the current slab. The statement of the time-discontinuous Galerkin method for the multi-field formulation is: Within each space-time slab, $n = 0, 1, ..., N-1$; the objective is to find $U_f^h := \{\phi^h\,,\,p^h\} \in \mathcal{T}_1^h \times \mathcal{T}_2^h$ and $U_s^h := \{u_s^h\,,\,v_s^h\} \in \mathcal{S}_1^h \times \mathcal{S}_2^h$, such that for all weighting functions $\delta U_f^h := \{\delta\phi^h\,,\,\delta p\} \in \mathcal{T}_1^h \times \mathcal{T}_2^h$, and $\delta U_s^h := \{\delta u_s^h\,,\,\delta v_s^h\} \in \mathcal{S}_1^h \times \mathcal{S}_2^h$, the following coupled variational equation is satisfied:

$$\begin{aligned}
& B_f(\delta U_f^h\,,\,U_f^h)_n + B_s(\delta U_s^h\,,\,U_s^h)_n + B_\infty(\delta U_f^h\,,\,U_f^h)_n \\
&= (\delta p^h\,,\,v_s^h \cdot n)_{(\Upsilon_i)_n} - (\delta v_s^h \cdot n\,,\,p^h)_{(\Upsilon_i)_n} \qquad (15)
\end{aligned}$$

with

$$\begin{aligned}
B_f(\delta U_f^h\,,\,U_f^h)_n &:= (\delta p^h\,,\,K_f^{-1}\dot{p}^h)_{Q_n^f} - (\nabla\delta p^h\,,\,v^h)_{Q_n^f} + (\delta v^h\,,\,\mathcal{L}_f U_f^h)_{\tilde{Q}_n^f} \\
&+ (\delta p^h(t_n^+)\,,\,K_f^{-1}[\![p^h(t_n^+)]\!])_{\Omega_f} + (\delta v^h(t_n^+)\,,\,\rho_f[\![v^h(t_n^+)]\!])_{\Omega_f} \qquad (16)
\end{aligned}$$

$$\begin{aligned}
B_s(\delta U_s^h\,,\,U_s^h)_n &:= (\delta v_s^h\,,\,\rho_s\dot{v}_s^h)_{Q_n^s} + a(\delta v_s^h\,,\,u_s^h)_{Q_n^s} + a(\delta u_s^h\,,\,\mathcal{L}_s U_s^h)_{\tilde{Q}_n^s} \\
&+ (\delta v_s^h(t_n^+)\,,\,\rho_s[\![v_s^h(t_n^+)]\!])_{\Omega_s} + a(\delta u_s^h(t_n^+)\,,\,[\![u_s^h(t_n^+)]\!])_{\Omega_s} \qquad (17)
\end{aligned}$$

in which $v^h = \nabla\phi^h$, $\delta v^h = \nabla\delta\phi^h$, and

$$\mathcal{L}_f U_f^h = \rho_f \dot{v}^h + \nabla p^h, \quad \text{and} \quad \mathcal{L}_s U_s^h = \dot{u}_s^h - v_s^h \qquad (18)$$

In the above expressions, a tilde refers to integration over element interiors. $B_f(\cdot\,,\,\cdot)_n$ and $B_s(\cdot\,,\,\cdot)_n$ are bilinear forms for the fluid and structure respectively. Fluid-structure interaction is accomplished through the coupling operators defined on the fluid-structure interface $(\Upsilon_i)_n := \Gamma_i \times I_n$. The operator B_∞, incorporates the time-dependent radiation boundary conditions on the fluid truncation boundary Γ_∞. The definition of this operator depends on the order of the spatial and/or temporal derivatives appearing in the radiation boundary condition. For example, for the local second-order boundary condition (14) the boundary operator is defined as,

$$\begin{aligned} B_\infty(\delta U_f^h\,,\,U_f^h)_n &= d_2(\delta p^h\,,\,\dot{p}^h)_{(\Upsilon_\infty)_n} + d_1(\delta p^h\,,\,p^h)_{(\Upsilon_\infty)_n} \\ &- d_0(\delta p^h\,,\,\phi^h)_{(\Upsilon_\infty)_n} + d_0(\delta \phi^h\,,\,p^h + \rho_f \dot{\phi}^h)_{(\Upsilon_\infty)_n} \\ &+ d_2(\delta p^h(t_n^+)\,,\,[\![p^h(t_n)]\!])_{(\Upsilon_\infty)_n} + d_0(\delta \phi^h(t_n^+)\,,\,\rho_f [\![\phi^h(t_n)]\!])_{(\Upsilon_\infty)_n} \end{aligned} \tag{19}$$

where

$$\begin{aligned} d_0(\delta p^h\,,\,\phi^h)_{(\Upsilon_\infty)_n} &:= \frac{1}{R}(\delta p^h\,,\,\phi^h)_{(\Upsilon_\infty)_n} + \frac{1}{2R}(\delta p^h_{,\varphi}\,,\,\phi^h_{,\varphi})_{(\Upsilon_\infty)_n} \\ &+ \frac{1}{2R}(\delta p^h_{,\theta}\,,\,\csc^2(\varphi)\,\phi^h_{,\theta})_{(\Upsilon_\infty)_n} \end{aligned} \tag{20}$$

$$d_1(\delta p^h\,,\,p^h)_{(\Upsilon_\infty)_n} := \frac{2}{\rho_f c}(\delta p^h\,,\,p^h)_{(\Upsilon_\infty)_n} + \frac{R}{\rho_f c}(\delta p^h\,,\,p^h_{,r})_{(\Upsilon_\infty)_n} \tag{21}$$

$$d_2(\delta p^h\,,\,\dot{p}^h)_{(\Upsilon_\infty)_n} := \frac{R}{K_f}(\delta p^h\,,\,\dot{p}^h)_{(\Upsilon_\infty)_n} \tag{22}$$

In the above, integration-by-parts has been used to relax the continuity implied by the second-order tangential derivatives appearing in Δ_Γ from $C^1(\Gamma_\infty)$ to $C^0(\Gamma_\infty)$. The form of the terms defined in (19) involving temporal jump operators evaluated on the boundary Γ_∞, can be inferred from (20) and (22). These consistent jump terms act to weakly enforce continuity of U_f^h between space-time-slabs at the boundary Γ_∞. These additional operators are needed in order to ensure unconditional stability for the solution and are the crucial element that enable generalization of the time-discontinuous space-time finite element method to handle unbounded domains.

A system of algebraic equations is obtained by introducing space-time finite element approximations for the independent variables:

$$\phi^h(x\,,\,t) = N_f(x\,,\,t)\phi, \qquad (x\,,\,t) \in Q_n^f \tag{23}$$

$$p^h(x\,,\,t) = \chi_f(x\,,\,t)p, \qquad (x\,,\,t) \in Q_n^f \tag{24}$$

$$u_s^h(x\,,\,t) = N_s(x\,,\,t)d, \qquad (x\,,\,t) \in Q_n^s \tag{25}$$

$$v_s^h(x\,,\,t) = \chi_s(x\,,\,t)c, \qquad (x\,,\,t) \in Q_n^s \tag{26}$$

with their associated weighting (variational) parameters. In these expressions $\{N_f\,,\,N_s\} \in \mathcal{T}_1^h \times \mathcal{S}_1^h$ and $\{\chi_f\,,\,\chi_s\} \in \mathcal{T}_2^h \times \mathcal{S}_2^h$ are arrays defining global basis functions over a space-time slab, and $\{\phi\,,\,p\}$ and $\{d\,,\,c\}$ are global solution vectors. Inserting (23) – (26) into the variational equation (15) leads to the coupled system of algebraic equations to be solved in sequence for each time interval $I_n =]t_n\,,\, t_{n+1}[,\;\; n = 0, 1, \cdots N-1$:

$$\begin{bmatrix} K_f & C_f & 0 & 0 \\ C_f^T & M_f & A & 0 \\ 0 & A^T & M_s & C_s \\ 0 & 0 & C_s^T & K_s \end{bmatrix} \begin{Bmatrix} \phi \\ p \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} f_f^1 \\ f_f^2 \\ f_s^2 \\ f_s^1 \end{Bmatrix} \tag{27}$$

where K_s, M_s are global matrices emanating from the structural operator B_s, and K_f, M_f are global matrices emanating from the fluid operator B_f and B_∞; C_f is the coupling matrix relating acoustic pressure and velocity potential solution arrays; likewise C_s is the coupling matrix relating structural displacement and velocity degrees-of-freedom; A is the fluid-structure coupling matrix defined as:

$$A^T \;=\; \int_{t_n}^{t_{n+1}} \int_{\Gamma_i} \chi_s^T\, n\, \chi_f \; d\Gamma\; dt \tag{28}$$

Stability and Accuracy Results

The positive form of (27) follows directly from a stability (coercivity) result derived in [9]:

$$\mathbb{E}(t_{n+1}^-) + B_\infty(U_f^h\,,\, U_f^h)_n \le \mathbb{E}(t_n^-), \quad \forall\, \Delta t_n > 0, \tag{29}$$

and $n = 0, 1, \ldots N-1$. Eq. (29) states that the computed total energy for the system

$$\mathbb{E}(U_f^h\,,\, U_s^h) := \mathcal{E}_f(U_f^h) + \mathcal{E}_s(U_s^h) \tag{30}$$

$$\mathcal{E}_s(U_s^h) \;=\; \frac{1}{2}\,(v_s^h\,,\, \rho_s v_s^h)_{\Omega_s} + \frac{1}{2}\, a(u_s^h\,,\, u_s^h)_{\Omega_s} \tag{31}$$

$$\mathcal{E}_f(U_f^h) \;=\; \frac{1}{2}\, \|K_f^{-1/2} p^h\|_{\Omega_f}^2 + \frac{1}{2}\, \|\rho_f^{1/2} v^h\|_{\Omega_f}^2 \tag{32}$$

plus the radiation energy absorbed through the artificial boundary $B_\infty(U_f^h\,,\, U_f^h)$ at the end of a time step is always less than or equal to the total energy at the previous time step for arbitrary step sizes. This result implies that *the space-time formulation presented is unconditionally stable.*

For additional stability, local residuals of the governing differential equations in the form of least-squares may be added to the Galerkin variational equations. Stabilized methods of this type are referred to as Galerkin Least Squares (GLS) methods, and in the context of transient wave propagation, may be designed to provide numerical dissipation of unresolved high-frequencies without degrading the accuracy of the underlying Galerkin method. GLS methods have also been used to improve the accuracy for the related reduced wave equation (Helmholtz equation) governing time-harmonic acoustics in the frequency domain, [12, 15, 16].

If the finite element approximation for the acoustic pressure is selected such that, $p^h = -\rho_f \dot{\phi}^h$, which implies that the residual $\mathcal{L}_f U_f^h = \rho_f \nabla \dot{\phi}^h + \nabla p^h = 0$, and similarly if the structural velocity is the time derivative of the structural displacement, $v_s^h = \dot{u}_s^h$, then the multi-field formulation (15) specializes to the single-field formulation presented in [4, 5, 6]. This simplification occurs when the temporal order of approximation for p^h and v_s^h is one order less than that used for ϕ^h and u_s^h, respectively; i.e., $\{\chi_f\,,\, \chi_s\} = \{N_{f,t}\,,\, N_{s,t}\}$. For the single field formulation, quadratic interpolation in the time-dimension is required to resolve the second-order time derivatives appearing on ϕ^h and u_s^h in the simplified variational equation.

As a result of being a weighted residual based formulation, the method presented is *consistent* in the sense that for a sufficiently smooth exact solution to the initial/boundary-value problem (1) – (6), then the error is orthogonal with respect to the variational operator (15). Consistency is necessary for maintaining optimal convergence rates for higher-order basis functions. For example, for approximations of the form $p^h = -\rho_f \dot{\phi}^h$ and $v_s^h = \dot{u}_s^h$, and the time-discontinuous GLS formulation, it has been shown in [5] that the approximation error $E = \{\phi^h - \phi\, ,\, u_s^h - u_s\}$, converges at the rate

$$|||E|||^2 \quad \leq \quad c(\phi)\, h_f^{2k-1} + c(u_s)\, h_s^{2m-1} \tag{33}$$

where $h_s = \max\{c_L \Delta t\, ,\, \Delta x\}$, and $h_f = \max\{c\Delta t\, ,\, \Delta x\}$ are element mesh size parameters, c_L is the dilatational wave speed in Ω_s and c is the acoustic wave speed in Ω_f; Δx and Δt are maximum element diameters in space and time, respectively; $c(u)$ and $c(\phi)$ are values that are independent of h_s, h_f. The integers k and m are the finite element interpolation orders for the fluid and structure respectively. The norm $|||E|||$ in which convergence is measured emanates naturally from the coupled fluid-structure variational equation together with additional least-squares operators. Equation (33) indicates that the error for the coupled system is controlled by the convergence rates in both the structure and the fluid; i.e., for an accurate solution to the coupled fluid-structure problem, discretizations for both the structural domain and the fluid domain must be adequately resolved. Accuracy can be increased in both space and time by simply increasing the order of the polynomial used in the finite element approximation.

Respresentative Numerical Example

In Figure 2, results are presented for the transient scattering from a rigid cylinder with conical-to-spherical end caps and a large length-to-diameter ratio. This example represents a challenging problem where the multiple-scales involving the ratio of the wavelength to cylinder diameter and cylinder length dimension play a critical role in the complexity of the resulting scattered wave field. For this example, quadratic interpolation is used for the approximation to ϕ^h and $p^h = -\rho_f \dot{\phi}^h$. On the truncation boundary Γ_∞, a local second-order non-reflecting boundary condition is applied. The numerical simulation starts with an initial pulse at $t = 3$. At $t = 6$ the incident pulse has expanded and has just reached the boundaries of the rigid cylinder. At the non-reflecting boundary Γ_∞, the wave front is allowed to pass through the boundary with negligible reflection. As time progresses, the wave has begun to reflect off the rigid boundary, creating a complicated backscattered wave, that eventually passes through Γ_∞, leaving a quiscent solution in its wake. This example illustrates the high-order accuracy achieved by the space-time method for acoustic scattering. Further details and a number of other examples may be found in [5, 7, 8].

Conclusions

The numerical solution of many practical problems of transient structural acoustics and fluid-solid interaction is still far from a reality. Innovative methods, such as the space-time finite element methods discussed in this paper are needed to advance the solution of large-scale problems involving structures submerged in an infinite acoustic region. In recent years, a time-discontinuous Galerkin formulation has evolved which is applicable to general coupled structural acoustics including high-order accurate non-reflecting boundary conditions. The resulting finite element methods are unconditionally stable and allow for unstructured meshes and high-order approximations in both space and time. In addition, the space-time finite element approach provides a nice framework for developing space-time adaptive schemes and subcyling schemes in which different time steps are used in different elements. The attributes of the methods not possessed by in-place procedures indicate that there is considerable potential for future applications in acoustics and wave propagation problems in general.

References

[1] T.J.R. Hughes and G.M. Hulbert. Space-time finite element methods for elastodynamics: Formulations and error estimates. *Comp. Methods in Applied Mech. Engng.*, 66:339–363, 1988.

[2] F. Shakib, T.J.R. Hughes, and Z. Johan. A new finite element formulation for computational fluid dynamics, X: The compressible Euler and Navier-Stokes equations. *Comp. Methods in Applied Mech. Engng.*, 89:141–219, 1991.

[3] S.K. Aliabadi and T.E. Tezduyar. Space-time finite element computation of compressible flows involving moving boundaries and interfaces. Technical Report 92/95, University of Minnesota Supercomputer Institute Research Report, April 1992.

[4] L.L. Thompson and P.M. Pinsky. New space-time finite element methods for fluid-structure interaction in exterior domains. In *Computational Methods for Fluid/Structure Interaction*, volume AMD-Vol. 178, pages 101–120. ASME, 1993.

[5] L.L. Thompson. *Design and Analysis of Space-time and Galerkin Least-Squares Finite Element Methods for Fluid-Structure Interaction in Exterior Domains.* PhD thesis, Stanford University, April 1994.

[6] L.L. Thompson and P.M. Pinsky. A space-time finite element method for structural acoustics in infinite domains, Part I: Formulation, stability, and convergence. *Comp. Methods in Applied Mech. Engng.*, Accepted: 1995.

[7] L.L. Thompson and P.M. Pinsky. A space-time finite element method for structural acoustics in infinite domains, Part II: Exact time-dependent non-reflecting boundary conditions. *Comp. Methods in Applied Mech. Engng.*, Accepted: 1995.

[8] L.L. Thompson and P.M. Pinsky. A space-time finite element method for the exterior structural acoustics problem: Time-dependent radiation boundary conditions in two spatial dimensions. *Int. J. Numer. Methods Engng.*, Accepted: 1995.

[9] L.L. Thompson. A multi-field space-time finite element method for structural acoustics. In *Proceedings of the symposium on acoustics of submerged structures and transduction systems.* ASME 15th Biennial Conference on Mechanical Vibration and Noise, Sept. 17-21, Boston, Mass., 1995.

[10] J.B. Keller and D. Givoli. Exact non-reflecting boundary conditions. *J. Comput. Phys.*, 82(1):172–192, 1989.

[11] D. Givoli and J.B. Keller. Non-reflecting boundary conditions for elastic waves. *Wave Motion*, 12:261–279, 1990.

[12] I. Harari. *Computational Methods for Problems of Acoustics with Particular Reference to Exterior Domains.* PhD thesis, Stanford University, 1991.

[13] A. Bayliss and E. Turkel. Radiation boundary conditions for wave-like equations. *Commun. Pure Appl. Math.*, 33:707–725, 1980.

[14] D. Givoli and J.B. Keller. Special finite elements for use with high-order boundary conditions. *Comp. Methods in Applied Mech. Engng.*, 119:199–213, 1994.

[15] L.L. Thompson and P.M. Pinsky. A multi-dimensional galerkin least-squares finite element method for time-harmonic wave propagation. In R. Kleinman et. al., editor, *Second International Conference on Mathematical and Numerical Aspects of Wave Propagation*, pages 444–451. SIAM, 1993.

[16] L.L. Thompson and P.M. Pinsky. A Galerkin least squares finite element method for the two-dimensional Helmholtz equation. *Int. J. Numer. Methods Engng.*, 38:371–397, 1995.

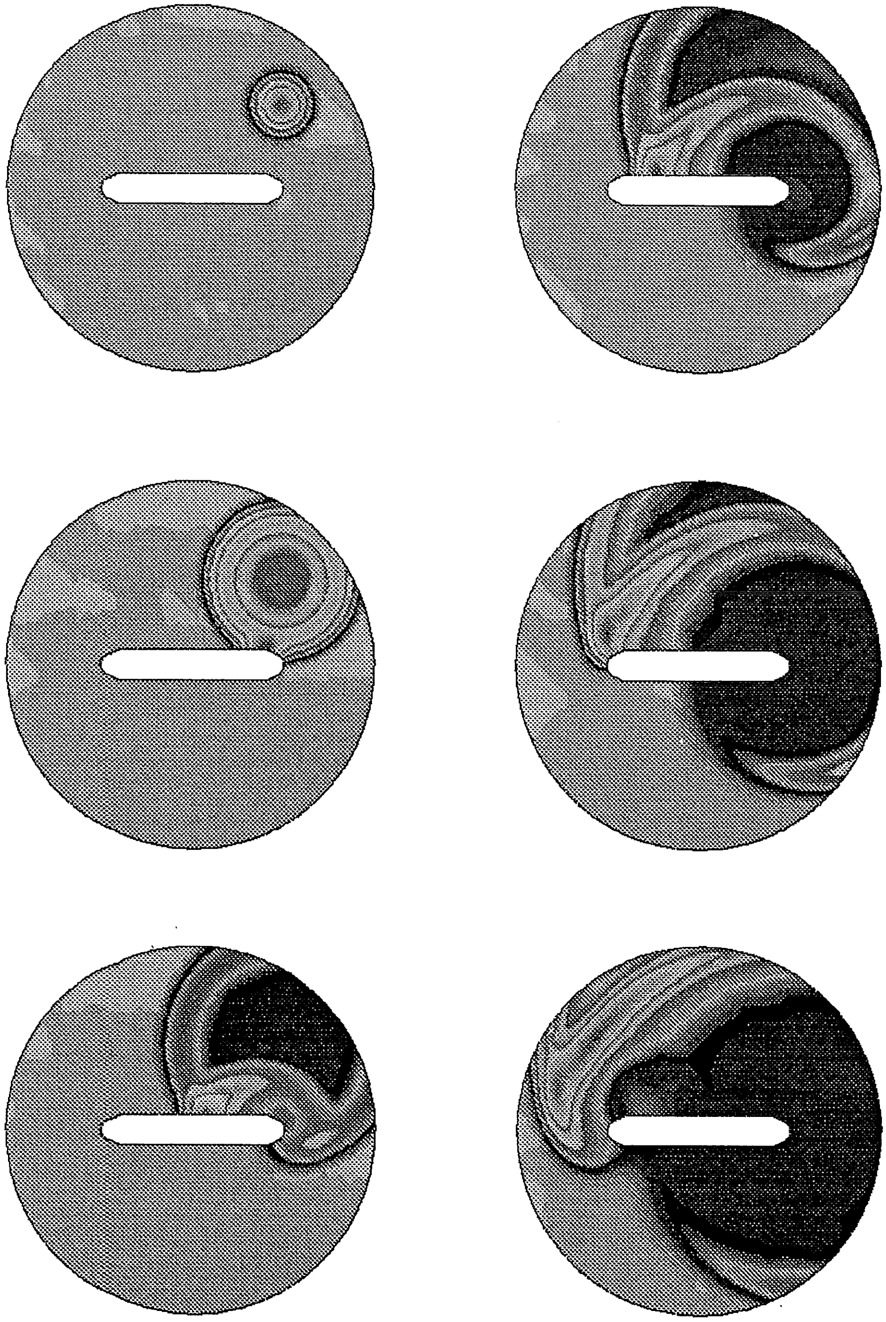

Figure 2: Scattering from a rigid cylinder due to a point source. Solution contours shown at the end of the initial pulse at $t = 3$ and later times $t = 6, 9, 12, 15, 18$.