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Acoustic Solitary Waves in a Tunnel with an Array of Helmholtz Resonators

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1. Introduction

It has recently been demonstrated theoretically that a soliton† is possible even in propagation of nonlinear acoustic waves through the air [1-3]. This acoustic soliton is predicted for propagation in a tunnel with a periodic array of Helmholtz resonators, if their natural frequency is set high enough and dissipative effects are made negligible. Since nonlinear acoustic waves evolve usually into shock waves, it is the new finding. Physical mechanism to bear the soliton is the dispersion of Bloch wave type due to the spatially periodic structure [4].

If the condition of the high natural frequency is relaxed, the soliton cannot generally exist but there still remains the possibility that other solitary waves can be propagated in place of it. This article clarifies the point by examining the steady propagation of nonlinear acoustic waves in the tunnel. Analysis is made on the basis of the equations derived previously under the continuum approximation for the periodic array of Helmholtz resonators [1].

2. Basic equations for nonlinear acoustic waves in the tunnel

Supposing the lossless and one-dimensional propagation along the tunnel, the basic equations are given as follows:

$$\frac{\partial f}{\partial X} - f\frac{\partial f}{\partial \theta} = -K\frac{\partial g}{\partial \theta}\ , \tag{1}$$

$$\frac{\partial^2 g}{\partial \theta^2} + \Omega g = \Omega f\ , \tag{2}$$

where f and g denote the dimensionless excess pressure appropriately normalized, respectively, in the tunnel and in the cavity of resonators, while X and θ denote, respectively, the dimensionless far-field axial coordinate and the retarded time in a frame moving with the linear sound speed. Here the parameters K and Ω measure the ratio of the smallness

†By a soliton, we mean a solitary wave as a steady-wave solution to the Korteweg-de Vries equation.

of the resonator κ to the small nonlinearity ε, and the ratio of the resonator's natural angular frequency ω_0 to a typical frequeny of the acoustic waves ω, defined, respectively, as follows:

$$K = \frac{\kappa}{2\varepsilon} \quad \text{and} \quad \Omega = \left(\frac{\omega_0}{\omega}\right)^2 , \tag{3}$$

where κ ($\ll 1$) is the ratio of the cavity's volume relative to the tunnel's volume per axial spacing between the neighboring resonators and ε ($\ll 1$) designates the order of the excess pressure relative to the atmospheric. Incidentally if Ω is large enough, g is approximated by (2) to be

$$g = f - \frac{1}{\Omega}\frac{\partial^2 g}{\partial \theta^2} = f - \frac{1}{\Omega}\frac{\partial^2 f}{\partial \theta^2} + O\left(\frac{1}{\Omega^2}\right) . \tag{4}$$

Substituting (4) into (1), we derive immediately the Korteweg-de Vries equation (called simply K-dV equation hereafter).

The parameters K and Ω can be removed from (1) and (2) by replacement of (f, g) and (X, θ), respectively, with (Kf, Kg) and $(X/K\sqrt{\Omega}, \theta/\sqrt{\Omega})$. Thus they are set equal to unity without any loss of generality in the following analysis.

3. Propagation of solitary waves

We seek steady-wave solutions to (1) and (2) by assuming that f and g depend on X and θ only through a combination $\theta - sX$ ($\equiv \zeta$), s being a parameter. Here we simply note that s is a parameter to measure the deviation of physical propagation velocity $a_0(1 - \varepsilon K s)$ from a_0. By imposing the boundary condition that the state far ahead of propagation is undisturbed, there exist the solutions satisfying the undisturbed condition far behind as well as far ahead. For $0 < s < 1$, in fact, we have for $0 \le f \le f_+$

$$4 \tan^{-1}\sqrt{\frac{f_+ - f}{f - f_-}} - \frac{2s}{\sqrt{-f_+ f_-}} \log\left|\frac{[\sqrt{-f_-(f_+ - f)} - \sqrt{f_+(f - f_-)}]^2}{(f_+ - f_-)f}\right| = \pm\zeta , \tag{5}$$

where f_+ and f_- ($-4/3 < f_- < 0 < f_+ < 8/3$) are constants determined by s and the ζ axis is chosen so that f takes the maximum value f_+ at $\zeta = 0$. When f is available, g is obtained by the relation $g = f^2/2 + sf$.

Figure 1 shows the explicit profiles of f for some values of s. As can be seen, they are localized in ζ (i.e. spatially and temporally) and represent the solitary waves. It is found that the solitary waves are *compressive* and that they are propagated with speed slower than the usual sound speed a_0 (i.e. *subsonic*), but faster than $a_0(1 - \kappa/2)$ in the linear long-wave limit. As the propagation speed approaches the upper bound ($s \to 0+$), the

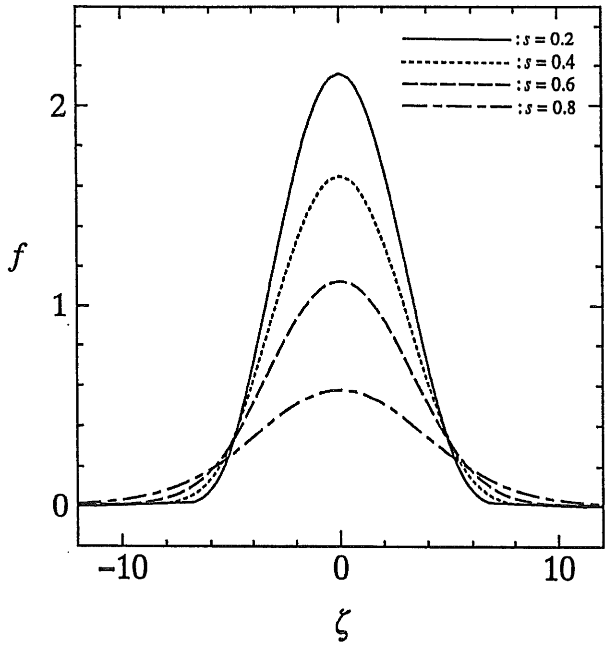

Figure 1: Profiles of the solitary-wave solutions f for $s = 0.2, 0.4, 0.6$ and 0.8 where K and Ω are set equal to unity.

solitary waves tend to the limiting solitary wave given by

$$f = \frac{8}{3}\cos^2\frac{\zeta}{4}\,, \tag{6}$$

where $-2\pi \leq \zeta \leq 2\pi$ and f vanishes in the outside of this interval, and therefore the height of the solitary waves approaches the limiting height 8/3. The excess pressure Δp corresponding to this height is given in reference to the atmospheric p_0 by

$$\frac{\Delta p}{p_0} = \frac{8\gamma}{3(\gamma+1)}\,\kappa\,, \tag{7}$$

where γ is the ratio of the specific heats.

As the speed approaches the lower bound ($s \to 1-$), on the other hand, the solitary waves tend to be the soliton solution of the K-dV equation [2,3]:

$$f = \alpha\,\mathrm{sech}^2\sqrt{\frac{\alpha}{12}}\zeta\,, \tag{8}$$

where α $(0 < \alpha \ll 1)$ is arbitrary and $s = 1 - \alpha/3$. Hence it is found that the soliton is included as the special case. For the speed below the lower bound, no steady-wave solutions exist, while for the speed above the upper bound (i.e. *supersonic*), the discontinuous shock waves are allowed in the solutions. Unlike usual shock waves in the air, there appears behind the discontinuity a nonlinearly oscillatory and periodic wavetrain extending downstream. As the speed increases, the discontinuity also increases but the amplitude of the wavetrain approaches a finite value, while as the speed decreases to approach the linear sound speed, the shock waves tend to diminish of course but the wavetrain remains.

4. Conclusions

It has been demonstrated that the acoustic solitary waves can be propagated in a tunnel with a periodic array of Helmholtz resonators. Their explicit profiles have been obtained analytically. It is revealed that while the height of the solitary waves is limited by the limiting solitary wave, solitary waves of small height are well described by the soliton. If a typical frequency of the solitary wave is defined as π/D, D being its temporal half width, it is found that the solitary waves are possible for $\Omega > 4$. In the case of the soliton, D becomes long in proportion to the inverse square root of the height. Therefore its typical frequency is so low that the assumption $\Omega \gg 1$ is satisfied.

References

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