Theoretical and Computational Acoustics
D. Lee, Y-H Pao, M.H. Schultz, and Y-C Teng (Editors)

Two-Dimensional Acoustic Wave Propagation in a Medium Containing Rigid Cracks

Guus J.H. Muijres and Gérard C. Herman
Faculty of Technical Mathematics and Computer Science
Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

Abstract

Wave propagation through media containing a large number of inclusions or cracks is a computationally intensive problem, especially if one is interested in dynamic properties of the medium like apparent absorption or dispersion due to multiple scattering caused by the inclusions. In the present paper, an integral-equation method is presented that is particularly efficient due to the use of appropriately chosen expansion functions. In a number of model studies, the method is compared with a perturbative solution and it is found that the latter can also be accurate provided the inclusions are small enough. For the special case of a monochromatic incident plane wave, the perturbative solution can then be used to replace the actual cracks by a much smoother 'apparent' medium which accounts for multiple-scattering effects in terms of a frequency-dependent and angle-dependent dispersion.

Introduction

The variations in the earth's subsurface range from scales much larger than seismic wavelengths down to scales that are much smaller. These small-scale variations can have a significant effect on the amplitude and phase of the transmitted wave field, as was shown for plane-stratified subsurface models [O'Doherty and Anstey, 1971]. Both stochastic and deterministic methods have been developed for studying this transmission problem in more detail. The deterministic approaches developed so far (see, for instance, [Burridge et al., 1988]) are mainly limited to plane-stratified models. Present-day theories for the propagation of acoustic waves through media containing small-scale inclusions are mostly stochastic (see, for a recent overview, [Hudson and Knopoff, 1989]).
In this paper we discuss a fully deterministic method for computing the wave field transmitted through a 2-D medium containing a large number of small-scale rigid cracks. A rigid crack is characterized by a vanishing normal component of the pressure gradient at the cracks. For a similar treatment of compliant cracks and circular heterogeneities we refer to [Muijres and Herman, 1994].
Starting from an integral representation for the pressure, an integral equation is obtained

for the unknown jump in the pressure across the cracks. By choosing adequate expansion functions, an efficient set of equations is derived with only one unknown coefficient per crack. In this respect, the method compares favorably with methods based on discretization of the wave equation.

We have compared the numerical solution of the resulting system of equations to solutions based on a Neumann series expansion taking scattering processes up to second-order into account. We have found that this expansion is accurate provided the cracks are small enough. This Neumann series expansion can then be used to replace the actual heterogeneous medium by a much simpler 'apparent' one [Herman, 1994].

Formulation of the problem

We consider two-dimensional, acoustic scattering from a large number of small-scale rigid cracks, embedded in a homogeneous medium. The n^{th} crack, of width $2a_n$, occupies the region C_n:

$$C_n = \{ \ (x,z) \ : \ |x - x_n| < a_n \ \wedge z = z_n \ \}. \tag{1}$$

In Eq. (1), x_n and z_n are the horizontal and vertical coordinate of the center of the crack, respectively. All cracks are assumed to be horizontally aligned.

The total pressure field, p, can be written as a superposition of the incident field, p^{inc}, which is the field in the absence of cracks, and the scattered field, p^{sc}, which accounts for the presence of the cracks:

$$p(x,z;\omega) = p^{inc}(x,z;\omega) + p^{sc}(x,z;\omega), \tag{2}$$

where ω is the angular frequency. For brevity we omit the explicit ω-dependence in the remainder. Outside the cracks, p satisfies the Helmholtz equation

$$\nabla^2 \ p(x,z) + \frac{\omega^2}{c_0^2} \ p(x,z) = -s(x,z), \tag{3}$$

where c_0 is the velocity of the embedding medium and $s(x,z)$ is the source that generates the incident field. If the source is a point source (i.e. $s(x,z) = \delta(x - x', z - z')$) then the solution of Eq. (3) is the Green's function for the embedding medium, given by

$$p^G(x,z;x',z') = \frac{i}{4} H_0^{(1)}(k_0 r), \tag{4}$$

in which $H_0^{(1)}(k_0 r)$ is the zeroth order Hankel function of the first kind, r is given by

$$r(x,x',z,z') = \sqrt{(x-x')^2 + (z-z')^2}, \tag{5}$$

and k_0 is the wavenumber ($k_0 = \omega / c_0$). In the following, the Green's function is used for deriving an integral equation formulation from which the scattered field can be determined.

Integral equation for the rigid crack

The presence of rigid cracks is accounted for by the Neumann boundary condition:

$$\forall_n \quad \lim_{z \to z_n} \frac{\partial p}{\partial z}(x,z) = 0, \quad |x - x_n| < a_n. \tag{6}$$

From the Helmholtz Eq. (3), the following integral representation can be derived for the scattered field outside the crack [van den Berg, 1981]:

$$p^{sc}(x,z) = \sum_{n=1}^{N} \int_{x_n - a_n}^{x_n + a_n} dx' \, \frac{\partial p^G}{\partial z'}(x, z; x', z'{=}z_n) \, \phi_n(x'), \quad (x,z) \notin C_n, \; n = 1, \cdots, N \tag{7}$$

where N denotes the number of cracks and ϕ_n represents the jump in the pressure across the crack,

$$\phi_n(x) = \lim_{z \downarrow z_n} p(x,z) - \lim_{z \uparrow z_n} p(x,z), \quad |x - x_n| < a_n. \tag{8}$$

From Eq. (7) and (8) it is seen that the jump ϕ_n in field at the cracks has to be known to compute the field outside the cracks. In order to determine ϕ_n, first a Fredholm integral equation of the first kind has to be derived from Eq.(7) [van den Berg, 1981]. To this aim, we take the partial derivative of Eq. (7) with respect to z and let the point of observation (x,z) approach crack C_m. With the aid of boundary condition (6), we then obtain

$$\forall_m \quad \frac{\partial p^{inc}}{\partial z}(x, z{=}z_m) = -\lim_{z \to z_m} \sum_{n=1}^{N} \int_{x_n - a_n}^{x_n + a_n} dx' \, \frac{\partial^2 p^G}{\partial z \partial z'}(x, z; x', z'{=}z_n) \, \phi_n(x'), \quad |x - x_m| < a_m. \tag{9}$$

Eq. (9) is a Fredholm integral equation of the first kind in the unknown functions ϕ_n.

Discretisation

In order to solve the integral in Eq.(9), ϕ_n is expanded in terms of an appropriately chosen sequence of functions. As the size of the cracks is much smaller than the wavelength, these expansion functions are chosen such that ϕ_n is accurately represented by only one expansion function per crack. This implies that we have

$$\phi_n(x) = b_n \, \psi_n(x), \quad |x - x_n| < a_n \, . \tag{10}$$

The choice for ψ_n will be specified later on.

To obtain a linear system of equations we multiply Eq. (9) with an appropriately chosen weight function $w_m(x)$, and integrate the result over C_m, to arrive at

$$\forall_m \quad \left(\frac{\partial p^{inc}}{\partial z}\right)_m = \sum_{n=1}^{N} G_{mn} \, b_n \tag{11}$$

where

$$\left(\frac{\partial p^{inc}}{\partial z}\right)_m = \int_{x_m - a_m}^{x_m + a_m} dx \; w_m(x) \, \frac{\partial p^{inc}}{\partial z}(x, z_m) \, , \tag{11a}$$

and

$$G_{mn} = -\lim_{z\to z_m} \int_{x_m-a_m}^{x_m+a_m} dx\; w_m(x) \int_{x_n-a_n}^{x_n+a_n} dx'\; \frac{\partial^2 p^G}{\partial z \partial z'}(x,z;x',z'{=}z_n)\; \psi_n(x')\,. \tag{11b}$$

Once this $N \times N$ linear system is solved the solutions b_n can be used to compute the field outside the cracks. Methods for solving Eq. (11) include explicit solvers for smaller values of N (< 500) and iterative schemes for larger crack numbers.

Choices for weight and expansion functions

For both the expansion and weight functions we make the following choice:

$$\begin{aligned} \psi_n(x) &= \sqrt{2}\,\sqrt{a_n^2 - (x-x_n)^2} \\ w_n(x) &= \sqrt{2}\,\sqrt{a_n^2 - (x-x_n)^2} \end{aligned}\;, \tag{12}$$

The motivation for this choice is that, for a single crack, it results in the same leading-order term as for the case of scattering by a small crack or slit [de Hoop, 1955b]. This implies that one expansion function is sufficient to represent the wave field provided the crack is small enough. With choices (12) the matrix elements of the kernel function $\frac{\partial^2 p^G}{\partial z \partial z'}$ are calculated in Appendix A and are given by:

$$G_{mn} = \begin{cases} \frac{\pi a_n^2}{2}\left(1 + \left(\frac{k_0 a_n}{2}\right)^2 \left(\log\left(\frac{k_0 a_n \gamma_e}{4}\right) - \frac{i\pi}{2} - \frac{3}{4}\right)\right) & m = n \\ -i\frac{\pi^2 a_m^2 a_n^2}{8} H_{mn}, & m \neq n \end{cases}\;, \tag{13}$$

where H_{mn} is given by

$$\begin{aligned} H_{mn} \;=\; \frac{k_0}{r_{mn}^3} \Big\{ & (z_m - z_n)^2\, k_0\, |x_m - x_n|\; H_0^{(1)}(k_0 r_{mn}) + \\ & \left((x_m - x_n)^2 + (z_m - z_n)^2\right) H_1^{(1)}(k_0 r_{mn}) \Big\}. \end{aligned} \tag{14}$$

Here $H_1^{(1)}$ is the first order Hankel function of the first kind and

$$r_{mn} = r(x_m, x_n, z_m, z_n). \tag{15}$$

Using Eq. (10) in Eq. (7), the following expression for the pressure outside the cracks results:

$$p(x,z) = p^{inc}(x,z) + \sum_{n=1}^{N} i\,\frac{\pi a_n^2}{4\sqrt{2}}\,k_0\,\frac{(z-z_n)}{r_n} H_1^{(1)}(k_0 r_n)\; b_n, \tag{16}$$

where $r_n = r(x, x_n, z, z_n)$. To compute the field, the solutions b_n from the system (11) are substituted. Eq. (16) shows an angle-dependent scattering because a rigid crack has an acoustic dipole radiation pattern.

Neumann series expansion

An alternative approach for solving Eq.(11) is the expansion of b_n in terms of a Neumann series (see, for instance, [Courant and Hilbert, 1931] (p. 119). The use of this approximate solution is motivated by the observation that, under the assumption of a monochromatic incident plane wave and periodically ordered cracks, this Neumann series expansion, if convergent, can be used to replace the actual cracked medium by an much simpler 'apparent' medium which accounts for multiple-scattering effects in terms of a frequency-dependent and angle-dependent dispersion and attenuation [Herman, 1994]. The accuracy of the Neumann series (and also of the apparent medium) can be checked against the numerical solution discussed in the previous section. To first order, this Neumann expansion for the coefficients b_m is given by

$$b_m = \frac{1}{G_{mm}}\Big(\frac{\partial p^{inc}}{\partial z}\Big)_m - \frac{1}{G_{mm}} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{G_{mn}}{G_{nn}}\Big(\frac{\partial p^{inc}}{\partial z}\Big)_n. \tag{17}$$

To compute the pressure outside the cracks these coefficients are substituted in Eq. (16). This yields

$$\begin{aligned} p(x,z) &= p^{inc}(x,z) - \sum_{n=1}^{N} \frac{1}{G_{nn}} \Big(\frac{\partial p^G}{\partial z}(x,z)\Big)_n \Big(\frac{\partial p^{inc}}{\partial z}\Big)_n \\ &+ \sum_{\substack{m,n=1 \\ m \neq n}}^{N} \frac{1}{G_{nn}} \Big(\frac{\partial p^G}{\partial z}(x,z)\Big)_n \frac{G_{nm}}{G_{mm}} \Big(\frac{\partial p^{inc}}{\partial z}\Big)_m, \end{aligned} \tag{18}$$

where $\Big(\frac{\partial p^G}{\partial z}(x,z)\Big)_n$ is given by

$$\Big(\frac{\partial p^G}{\partial z}(x,z)\Big)_n = \int_{x_n-a_n}^{x_n+a_n} dx' \, \frac{\partial p^G(x,z;x',z'=z_n)}{\partial z'} \, \psi_n(x'). \tag{19}$$

In expression (18), we recognize, from left to right, three contributions to the total wave field corresponding to different scattering processes: zeroth-order (incident field), first-order (single scattering at all cracks) and a second-order (double scattering involving all pairs of different cracks). In contrast to the numerical solution which takes all reflections into account, the Neumann expansion we use, only accounts for scattering processes up to second order.

Examples

We present the computed wave field for two distinct cases. First, we consider scattering from a single crack in order to compare our method (I) with a different method (II) developed by [Thorbecke, 1991]. This method (II) is based on an integral equation for the case of a single, large crack, which is solved numerically with a preconditioned conjugate

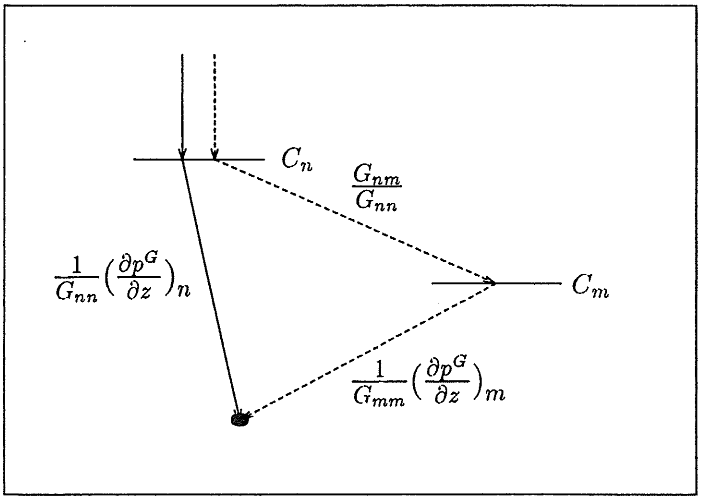

Figure 1: Physical interpretation of the Neumann series.

gradient scheme. The comparison is limited to intermediate crack-sizes as method (I) is most accurate for small cracks and method (II) for larger cracks. Therefore we choose a crack of half-width $a = 7.5\ m$, which is relatively large for our method (I), but relatively small for method (II). The configuration is shown in Fig. 2. The incident field is propagating in the negative z-direction. It has a dominant wavelength of about 50 m and it contains frequencies from 0 Hz to 40 Hz. The embedding velocity is 1500 m/s. The scattered field is computed for a receiver located 200 m above the origin. The results for both methods are shown together in Fig. 3. We observe a good agreement, especially if one realizes that the crack size is such, that both methods (I) and (II) have a limited accuracy. We now consider a plane wave propagating in the positive z-direction, through a medium containing 1000 randomly located cracks. The crack half-width is now chosen as $a = 1\ m$. The cracks are confined to a region of 250 $\times$ 250 m centered around the origin. The embedding velocity equals $c_0 = 3000\ m/s$. We calculate the transmitted field for a receiver at a depth of 400 m below the origin. The geometry is sketched in Fig. 4. The spectrum of the incident field contains frequencies between 5 Hz and 60 Hz and has a dominant wavelength of about 100 m. In Fig. 5 we observe that the presence of the cracks gives rise to dispersion of the waveform of the direct field.
To solve system (11) we have used a conjugate gradient method, in which all previous search directions are taken into account (see, for instance, [Vuik et al.]). Next we used the second order Neumann series expansion to compute the transmitted field at the receiver. Upon comparison of this solution and the previous we conclude that for small cracks ($a \leq 1\ m$) the Neumann solution is accurate. For larger cracks, the multiple-scattered energy is present in the coda and the Neumann solution is no longer accurate.
From further numerical experiments, we have found that, as the number of cracks increases, the transmitted field becomes more and more coherent and the main effect of the presence of the cracks seems to be an apparent dispersion. This suggests that it is possible to construct an apparent medium for media containing large numbers of rigid

cracks. This has already been shown by [Herman, 1994] for the case of small-scale velocity heterogeneities.

Conclusions

Based on an integral-equation formulation, an efficient forward modeling scheme is derived. For small cracks and carefully chosen expansion functions, each crack can be accurately accounted for by only one expansion function per crack. By using iterative techniques to solve for the unknown expansion coefficients, wave propagation through media containing large numbers of rigid cracks can be considered.
Comparisons of iterative solutions with solutions based upon a second-order Neumann series expansion show that the Neumann series is accurate provided the cracks are small enough. For the special case of a monochromatic incident plane wave and periodically ordered heterogeneities, this Neumann series expansion can then be used to replace the actual cracked medium by an much simpler 'apparent' medium which accounts for multiple-scattering effects in terms of a frequency-dependent and angle-dependent dispersion and attenuation [Herman, 1994]. This is the topic of our current research.

Appendix A: Calculation of the matrix elements G_{mn}

In the following we use the Fourier integral representation of the Hankel function $H_0^{(1)}$:

$$H_0^{(1)}\left(k_0\sqrt{(x-x')^2+(z-z')^2}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk_x \frac{e^{ik_x(x-x')+i\gamma(k_x)|z-z'|}}{\gamma(k_x)} \tag{20}$$

with $\gamma(k_x)^2 + k_x^2 = k_0^2$.
Partial derivatives with respect to z or z' are taken outside the integrals because they do not interfere with integrations over x or x'.
I. The diagonal elements ($m = n$).
In Eq. (11b) we use the integral representation Eq. (20). Then we apply twice the following property [van den Berg, 1981]

$$\int_{-a}^{a} dx\, \sqrt{1-(\frac{x}{a})^2}\, \exp(-ik_x x) = \pi\, \frac{J_1(k_x a)}{k_x} \tag{21}$$

to obtain

$$\begin{aligned} G_{nn} &= \lim_{\substack{z\to z_n \\ z'\to z_n}} -\frac{i\pi a_n^2}{2}\frac{\partial^2}{\partial z\partial z'}\left(\int_{-\infty}^{\infty} dk_x\, \frac{\exp(i\gamma(k_x)|z-z'|)}{k_x^2\gamma(k_x)}\, J_1^2(k_x a_n)\right) \\ &= -\frac{i\pi a_n^2}{2}\int_{-\infty}^{\infty} dk_x\, \frac{\gamma(k_x)\, J_1^2(k_x a_n)}{k_x^2} \equiv \pi a_n^2\, I(k_0 a_n). \end{aligned} \tag{22}$$

The last integral can be approximated to third order in $k_0 a$ [de Hoop, 1955a] by

$$\begin{aligned} I(k_0 a) &= \frac{1}{2} - \frac{1}{4\pi}\sum_{p=1}^{\infty} \frac{\Gamma(p+\frac{1}{2})\Gamma(p-\frac{1}{2})(-k_0 a^2)^p}{\Gamma(p)\Gamma(p+1)\Gamma(p+1)\Gamma(p+2)} \Big[2\log(k_0 a) - i\pi + \\ &\qquad \psi(p+\frac{1}{2}) + \psi(p-\frac{1}{2}) - \psi(p) - 2\psi(p+1) - \psi(p+2)\Big] \\ &= \frac{1}{2}\left(1 + \left(\frac{k_0 a}{2}\right)^2 \left(\log\left(\frac{k_0 a \gamma_e}{4}\right) - \frac{i\pi}{2} - 3/4\right)\right) + \mathcal{O}(k_0 a^4) \end{aligned}$$

where Γ and ψ are the gamma and psi functions (see [Abramowitz]) and $\log(\gamma_e) \approx 0.577215$ is Euler's constant. Using this result in Eq. (22) we get the following approximation for the diagonal elements

$$G_{nn} = \frac{\pi a_n^2}{2}\left(1 + \left(\frac{k_0 a_n}{2}\right)^2 \left(\log\left(\frac{k_0 a_n \gamma_e}{4}\right) - \frac{i\pi}{2} - \frac{3}{4}\right)\right). \tag{23}$$

II. The off-diagonal elements ($m \neq n$).

We perform a Taylor series expansion of $H_0^{(1)}(k_0|r - r'|)$ in x and x' around x_m and x_n, respectively. To lowest order this yields

$$H_0^{(1)}(k_0\sqrt{(x-x')^2 + (z-z')^2}\,) \approx H_0^{(1)}(k_0\sqrt{(x_m - x_n)^2 + (z-z')^2}\,). \tag{24}$$

After substitution in Eq. (11b) and calculating the resulting integrals we find for the off-diagonal elements

$$G_{mn} = \lim_{\substack{z \to z_m \\ z' \to z_n}} -\frac{i\pi^2 a_m^2 a_n^2}{8} \frac{\partial^2}{\partial z \partial z'} \left(H_0^{(1)}(k_0\sqrt{(x_m - x_n)^2 + (z-z')^2}\,)\right). \tag{25}$$

The partial derivatives can be calculated using [Abramowitz]

$$\frac{\partial}{\partial u} H_0^{(1)}(u) = -H_1^{(1)}(u), \tag{26}$$

which leads to the expressions (13) for the off-diagonal elements.

Appendix B: Calculation of the vector elements $\left(\frac{\partial p^{inc}}{\partial z}\right)_m$

We expand $\frac{\partial p^{inc}}{\partial z}(x, z_m)$ to lowest order in a Taylor series around $x = x_m$ and obtain from Eq. (11a)

$$\left(\frac{\partial p^{inc}}{\partial z}\right)_m = \lim_{z \to z_m} \frac{\pi a_m^2}{\sqrt{2}} \frac{\partial p^{inc}}{\partial z}(x_m, z). \tag{27}$$

The partial derivative can be calculated once the incident field is specified.

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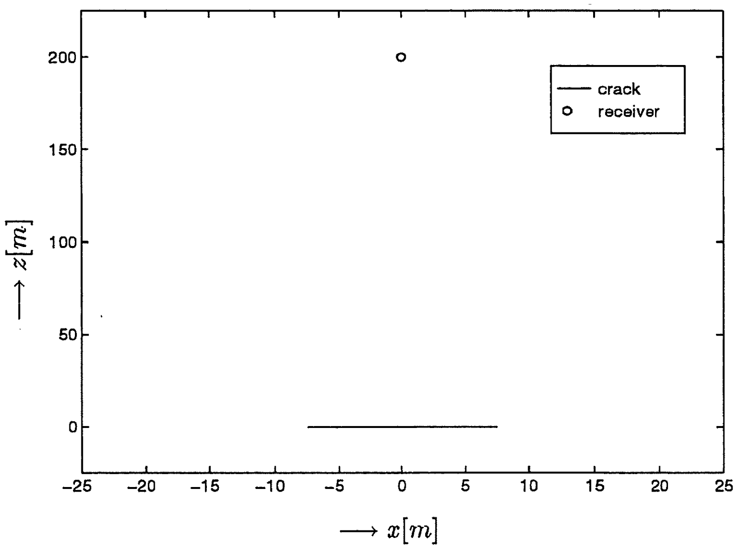

Figure 2: Crack ($a = 7.5\ m$) and receiver location $(0, 200)$ indicated with o.

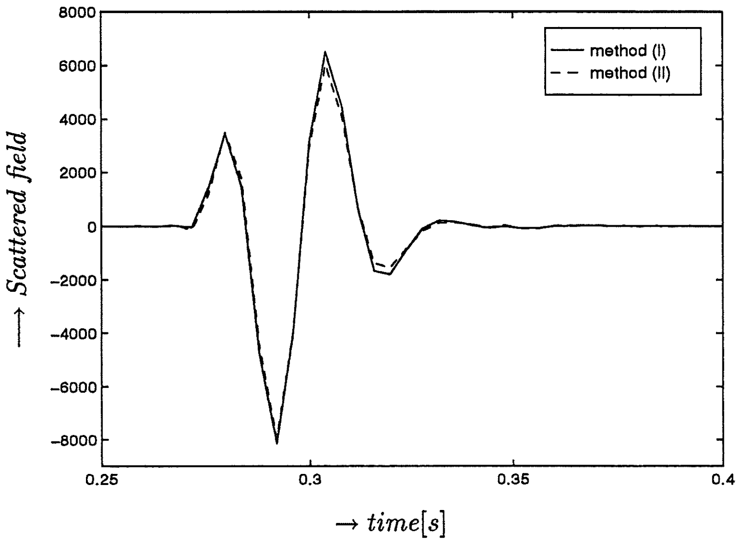

Figure 3: Comparison of scattered field from a single crack for 2 different methods (I) and (II) (See text). Crack size: $a = 7.5\ m$.

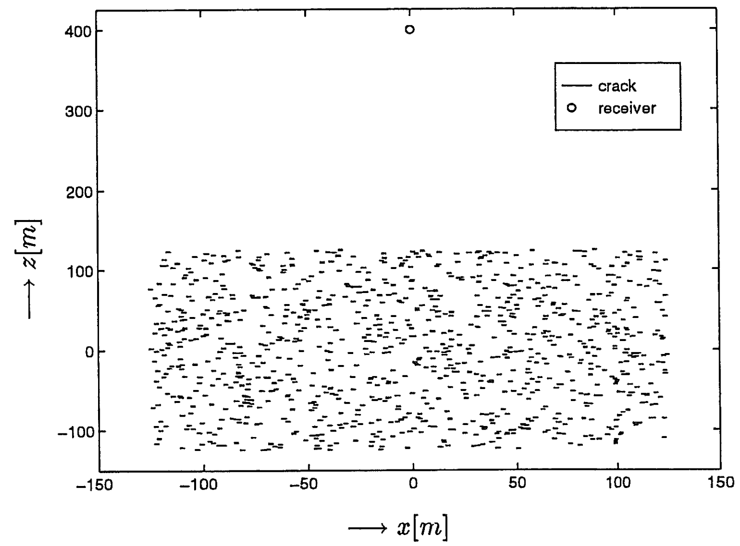

Figure 4: Crack distribution (1000 cracks, $a = 1\ m$) and receiver location $(0, 400)$ indicated with o.

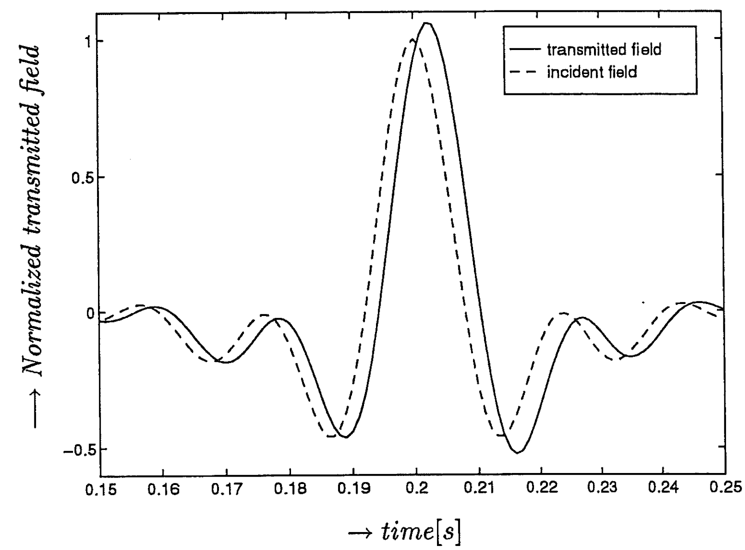

Figure 5: Comparison of the incident field with the transmitted field. Number of cracks: $N = 1000$, crack size: $a = 1\ m$.