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Higher Order Numerical Schemes for Paraxial Approximations of the Wave Equation

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Abstract

In a previous work, Collino and Joly [9] have constructed new 3D paraxial approximations that are compatible with the use of splitting methods without loss of accuracy. We present here new higher order numerical schemes to solve these equations in an heterogeneous medium. The dispersion effects observed by using classical second-order schemes can be considerably attenuated, even with a very few number of discretization points per wavelength. We also show how to adapt the method developed by Berenger for Maxwell equations in order to get absorbing layers on the lateral sides of the domain.

1 Introduction

Paraxial approximations constitute a good approximation of the wave equations when the waves propagate around a privileged direction. In many applications, one of the variable appears naturally as the privileged direction (z in geophysics, [6], r in ocean acoustics, [19]) and becomes an evolution variable. In the following, as we are mainly interested in geophysical applications, the evolution variable is the depth z. The classical way to discretize these equations consists in using a Crank-Nicolson scheme in the depth direction and second order finite differences for the derivatives in the lateral variables. In the 3D case, each extrapolation step requires the inversion of quite difficult and expensive linear systems (see Joly and Kern [13] and Kern [14]). This can be avoided by using the new higher order paraxial approximations constructed by Collino and Joly [9], which are compatible with splitting methods. The novelty of their approach in comparison with classical alternate direction methods ([5, 11, 17, 10]) is to introduce other directions for the splitting than the usual cross-line and in-line directions which allow to get forty-five degree and sixty-degree approximations. It reduces the problem to a series of 2D extrapolations in each direction of splitting. They have presented these new equations for constant velocities, but there is no difficulty to extend them to the case of heterogeneous velocities, following the criteria given by Bamberger and al. [2]. The observations of results obtained with the classical second-order numerical schemes have pointed out some dispersion effects of these numerical schemes, especially for large angles of propagation (with respect to the depth direction) and we propose here some new higher order numerical schemes that attenuate these effects.

The aim of this paper is to describe a systematic way to get accurate discretizations, both in depth and in the lateral variables to solve the 3D paraxial equations with splitting methods, in heterogeneous media.

The paper is organized as follows. In the first section we make some brief recalls on the classical paraxial approximations and set the PDEs associated to the new paraxial approximations. We also show how the method developed by Berenger [4] for Maxwell equations can be adapted for the paraxial equations, to achieve a quasi-perfect absorption of the waves on the lateral boundaries.

In section 3, we explain how to get higher order discretizations in the depth variable with the procedure presented by Kern (see [14]) for the full 2D paraxial equations based on a conservative Runge-Kutta method. Section 4 presents the discretization in the lateral variable with variational finite differences techniques. With these techniques, the variable coefficients are easily taken into account. The classical way to improve the accuracy is to use higher order discretizations of the derivatives in the lateral variables. However, this increases quickly the band-width of the linear system to be solved. We present a family of "modified" schemes of order $2n$ that cost the same price than the classical $(2n-2)$ order scheme. Numerical experiments show how the dispersion is attenuated with higher order schemes, especially with the modified schemes.

2 The continuous problem

2.1 The classical paraxial equations

Paraxial equations are approximations of the one way up-going wave equation

$$\frac{d\hat{v}}{dz} + i\frac{\omega}{c}\left(1 - \frac{c^2|k|^2}{\omega^2}\right)^{\frac{1}{2}}\hat{v} = 0, \tag{1}$$

where $\hat{v}$ denotes the Fourier transform of v with respect to t (time) and x_1, x_2. They are obtained by replacing the square root in (1) with rational fractions so as to transform this non local integro-differential equation into a local Partial Differential Equation, this approximation being valid as long as $c|k|/\omega$ remains small enough, i.e. as long as the wave propagates close enough to the z-direction. A general class of well-posed high order approximations has been proposed by Bamberger and al [1] and leads to the following system of (L+1) coupled PDEs , written in the frequency domain (since paraxial equations are usually handled in the frequency domain)

$$\begin{cases} \dfrac{\partial v}{\partial z} + \dfrac{i\omega}{c}v - \dfrac{i\omega}{c}\displaystyle\sum_{\ell=1}^{L} b^\ell \varphi_\ell = 0 \\ \dfrac{\omega^2}{c^2}\varphi_\ell + a^\ell \Delta\varphi_\ell = -\Delta v \qquad \ell = 1, \ldots, L. \end{cases}$$

Here L specifies the degree of the approximation and φ_ℓ are auxiliary unknown functions introduced so as not to deal with a high-order PDE.

Classically, we handle the transport term exactly, with the Claerbout change of unknown functions $u = ve^{i\omega z/c}$, and after elimination of the auxiliary unknown functions, end up with the

following system (written in operator form)

$$\begin{cases} \dfrac{\partial u}{\partial z} - i\dfrac{\omega}{c}\displaystyle\sum_{\ell=1}^{L} A^\ell u = 0 \\ A^\ell = -b^\ell (a^\ell \Delta + \dfrac{\omega^2}{c^2})^{-1}\Delta. \end{cases} \tag{2}$$

Since the evolution operator is written as a sum of simpler operators, it is a natural idea to use a splitting method to solve (2). Section 2.4 describes the algorithm in more details. Let us just mention here that, after discretization in depth, we have to solve, at each depth step, a linear system, with a large, sparse, complex valued, non-hermitian matrix. Kern [16] has proposed to use modern iterative methods to solve it. We investigate here an alternative way to avoid the difficulty altogether, by introducing a new class of paraxial equations suitable for use with splitting in the lateral variables, and requiring only the solution of 1D PDEs.

2.2 New paraxial approximations

Using splitting in the horizontal variables is not a new idea. In order to avoid the inversion of the large matrix when using the full paraxial approximation, several authors [5, 11, 17, 10] have advocated approximating the square root with

$$(1-|\kappa|^2)^{\frac{1}{2}} \approx 1 - \frac{1}{2}\frac{\kappa_1^2}{1-\frac{1}{4}\kappa_1^2} - \frac{1}{2}\frac{\kappa_2^2}{1-\frac{1}{4}\kappa_2^2} \quad (\text{ error : } O(\kappa_1^2\kappa_2^2)),$$

which is consistent with the forty-five degree equation (i.e., with an error $\approx O(|\kappa|^6)$) only in the $\kappa_1 = 0$ and $\kappa_2 = 0$ directions.

The basic idea to get better accurate approximations involving only one variable per fraction is to introduce more than two directions of splitting. The paraxial equations constructed in Collino and Joly [9] are derived from an approximation of the square root with rational fractions of the following form

$$(1-|\kappa|^2)^{1/2} = (1-(\kappa_1^2+\kappa_2^2))^{1/2} \approx 1 - R(\kappa_1,\kappa_2),$$

with

$$R(\kappa) = \sum_{j=1}^{N_D}\sum_{\ell=1}^{L} \frac{b_j^\ell(\kappa.n_j)^2}{1-a_j^\ell(\kappa.n_j)^2}, \tag{3}$$

where N_D corresponds to the number of directions, L the number of fractions per direction, and n_j the unit vector associated to the j^{th} direction ($n_j = (\cos\alpha_j, \sin\alpha_j)$). It has been shown in [9] that the conditions on the coefficients $a_j^\ell > 0$; $b_j^\ell \geq 0$ ensures the well-posedness of these paraxial equations. Also, in order to keep the approximation error in the square root from blowing-up in certain directions inside the unit disk $\kappa_1^2+\kappa_2^2 \leq 1$, it is natural to impose $0 < a_j^\ell \leq 1$.

Collino and Joly [9] have constructed several families of approximations of the above type so as to achieve comparable accuracy to the classical forty-five (error $\approx O(|\kappa|^6)$) or sixty degree (error $\approx O(|\kappa|^8)$) approximations. In particular, they obtained a family of forty-five degree approximations depending on one parameter using four directions of splitting ($N_D = 4$) uniformly distributed - namely $x_1, x_2, x_1+x_2, x_1-x_2$- and one fraction per direction ($L = 1$). For instance, the choice

$a_j = 1/3, b_j = 1/4, i = 1, \ldots, 4$ gives them the "maxi-isotropic" forty-five degree approximation. More examples are given in the above paper.

The paraxial equation corresponding to (3) can now be written in a form analogous to (2):

$$\begin{cases} \dfrac{\partial u}{\partial z} - i\dfrac{\omega}{c}\displaystyle\sum_{j=1}^{N_D}\sum_{\ell=1}^{L} A_j^\ell u = 0 \\ A_j^\ell u = -b_j^\ell (\dfrac{\omega^2}{c^2} + a_j^\ell D_j^2)^{-1} D_j^2 u, \end{cases} \tag{4}$$

where $D_j = n_j \cdot \vec{\nabla}$ is the derivative in the j^{th} direction. Each of the operators A_j^ℓ in (4) only involves a one-dimensional differential operator. Thus this new family equation lends itself to a splitting method in the horizontal variables, but as mentioned before, the splitting has the advantage of being consistent with the forty-five degree equation (for a proper choice of coefficients).

In practice, a mesh being given in the (x_1, x_2) plane, there will be two main choices for N_D: $N_D = 4$ if the mesh is built from squares, and then the directions are given by the two coordinate axes and the two main diagonals, or $N_D = 3$ if the mesh is built from equilateral triangles, and the 3 directions are 60° apart. Using more than 4 directions permits to get higher accuracy but would imply additional difficulties for the discretization which are beyond the scope of this paper.

In the following, for the sake of simplicity, we only consider the forty-five degree approximation family with $N_D = 4$ directions and $L = 1$ fraction per direction and we denote the corresponding coefficients and operator by a_j, b_j and A_j.

2.3 Extension to heterogeneous media and design of absorbing boundary conditions

The extension to heterogeneous media as well as the treatment of the absorbing boundaries are simply obtained by using similar modifications on the operators.

Paraxial equations in heterogeneous media have been proposed and analyzed by Bamberger and al. [2]. Their approach was to define several criteria (both mathematical and geophysical), and to select among a general class of possible candidates the one that satisfied those criteria.

Their result gives a recipe which allows one to extend any paraxial equation to heterogeneous media. Thus equations (4) keep the same form, with a new definition for the operators A_j

$$\begin{cases} \dfrac{\partial u}{\partial z} - i\dfrac{\omega}{c}\displaystyle\sum_{j=1}^{N_D} A_j u = 0 \\ A_j u = -b_j (\dfrac{\omega^2}{c^2} + a_j \delta_j^c)^{-1} \delta_j^c \end{cases}, \tag{5}$$

and we have defined $\delta_j^c = \frac{1}{c} D_j (c D_j)$.

A problem of practical importance is the treatment of the lateral boundaries. It must be designed in such a way that the waves are absorbed when they reach the boundaries. Recently, Collino [8] proposed to adapt a novel technique, introduced for electromagnetism by Bérenger [4]. This technique consists in designing an absorbing layer model called perfectly matched layer (PML). It possesses the astonishing property to generate no reflexion at the interface between the

free media and the artificial lossy medium, and the reflected waves are only due to the discretization of the model. This property allows one to use a very high damping parameter inside the layer, and consequently a short layer length, while still achieving a quasi-perfect absorption of the waves.

Practically, the PML is very easy to implement : we replace the velocity c in the operators A_j and δ_j^c defined previously by $cd(x)$ with $d(x) = \frac{i\omega}{i\omega + c\sigma(x)}$ and $\sigma(x)$ is a positive function with support in the damped area. We close the system with a Dirichlet boundary condition at the end of the layer. The efficiency of the method has been assessed by numerical experiments : a quasi-perfect absorption has been obtained with only 4 extra-nodes at the boundary and an appropriate choice for σ (see figure 2).

2.4 Splitting

We briefly recall some classical points about splitting methods (see [18]) which are specifically designed for the solution of ODEs in the form (5). The exact solution of (5) satisfies

$$u(z+\Delta z) = \exp\Big(\int_z^{z+\Delta z} \frac{i\omega}{c} \sum_{j=1}^{N_D} A_j dz\Big)\, u(z)\ .$$

The splitting methods consist in approximating the exponential of the sum of operators with the product of exponentials

$$u_{ap}(z+\Delta z) \approx \exp\left(\int_z^{z+\Delta z} \frac{i\omega}{c} A_{N_D} dz\right) \times \cdots \times \exp\left(\int_z^{z+\Delta z} \frac{i\omega}{c} A_1 dz\right) u(z), \tag{6}$$

which is second order accurate with respect to the discretization in the z variable, in the general case of an heterogeneous medium (in such a case the operators A_j do not commute). Approximation (6) then leads naturally to define N_D intermediate unknowns $w^j, j = 1, \ldots, N_D$ satisfying:

$$\begin{cases} \dfrac{dw_j}{dz} - \dfrac{i\omega}{c} A_j w_j = 0 \\ w_j(0) = w_{j-1}(\Delta z) \equiv w^{j-1}\ . \end{cases} \tag{7}$$

Finally we have $u_{ap}(z+\Delta z) = w^{N_D}$.

Problem (7), to be solved at each step, is still an evolution problem in depth, but with a single operator. In the next section, the discretization in the evolution variable is performed in order to get a high order semi-discretized scheme.

3 Semi discretization in depth

We assume in the following the velocity to be independent on z between z and $z+\Delta z$ (for instance, $c(x_1, x_2, z) = c^n(x_1, x_2)$ for $z \in [z^n, z^{n+1}]$). The discretization in the depth variable used here has been initially proposed by Joly and Kern [13]. It is based on the fact that the exact solution of (7) satisfies $w^j = e^{i\frac{\omega}{c} A_j \Delta z} w^{j-1}$, and on the relationship between Runge-Kutta methods and Padé approximations to the exponential (see for instance [12]). In order to get conservative

schemes of order $2K$, the exponential is replaced with a Padé approximant on the following form

$$\exp(ix) \approx \prod_{k=1}^{K} \frac{1+r_k x}{1+\bar{r}_k x},$$

where r_k are appropriate complex numbers and $\bar{r}_k$ is the complex conjugate of r_k. The integration from 0 to Δz is then done formally as follows

$$w^j = \prod_{k=1}^{K} (I + \bar{r}_k \Delta z A_j)^{-1} (I + r_k \Delta z A_j) w^{j-1}.$$

This procedure leads us to define K intermediate unknowns w_k^{j-1} associated to each fraction, solutions of

$$(I + \bar{r}_k \Delta z A_j) w_{k+1}^{j-1} = (I + r_k \Delta z A_j) w_k^{j-1}.$$

We then set $w^j = w_K^{j-1}$.

The classical Crank-Nicolson second-order scheme is obtained for $K = 1$ and $r_1 = i/2$. Kern [15] has shown that this process can also be used to get non conservative schemes, which generalize the θ-scheme to higher order schemes.

Finally, one elementary step requires the solution of systems of the form

$$\left(\mathrm{I} + d_j(\omega)\delta_j^c\right) U = F\,.$$

There remains the task of discretizing the 1D operator with respect to the lateral variable. We devote next section to specifying how this approximation is achieved.

4 Discretization in the lateral variables

In view of section 2.4, we now take for simplicity one particular direction, which we denote by x. The domain Ω is an interval in $\mathbf{R}$ and for clarity we make the presentation with Dirichlet conditions (but there is no difficulty to do the same with the PMLs). Thus we have to solve

$$\begin{cases} \dfrac{\partial w}{\partial z} - \dfrac{i\omega}{c}\varphi = 0 \quad \text{in } \Omega \\ \dfrac{\omega^2}{c}\varphi + \dfrac{\partial}{\partial x}\left(c\dfrac{\partial}{\partial x}(a\varphi + bw)\right) = 0 \quad \text{in } \Omega \\ w \equiv \varphi \equiv 0 \quad \text{on } \partial\Omega. \end{cases} \tag{8}$$

We base the discretization of (8) on a variational formulation, which we recall below. This provides us with a systematic treatment of heterogeneous media, in a way that insures good numerical properties (i.e., convergence and stability). We set

$$(u,v) = \int_\Omega u\bar{v}\,dx \quad ; \quad m(u,v) = \int_\Omega \frac{1}{c} u\bar{v}\,dx \quad ; \quad k(u,v) = \int_\Omega c\frac{\partial u}{\partial x}\frac{\partial \bar{v}}{\partial x}\,dx$$

With these definitions, problem (8) admits the variational formulation

$$\text{Find } (w,\varphi) \text{ such that } \begin{cases} \dfrac{d}{dz}(w,\chi) - i\omega m(\varphi,\chi) = 0 \quad \forall \chi \\ \omega^2 m(\varphi,\psi) - k(a\varphi + bw,\psi) = 0 \quad \forall \psi. \end{cases}$$

The discretization is built upon a variational finite differences method. It consists in approximating the operators with operators of finite difference type. Let us introduce some notations. The domain is discretized by a regular grid ($x_i = i\Delta x$) and we define the shifted grid by the nodes ($x_{i+1/2} = (i+1/2)\Delta x = (i+1/2)h$). The approximation is then decomposed on the functions χ_i where χ_i is the characteristic function on $[x_{i-1/2}, x_{i+1/2}]$. Since the functions χ_i are constants per element, one can define the (diagonal) mass matrix as $M^\star_{ij} = m(\chi_i,\chi_j)$, but one has to give a meaning to the derivative terms in the stiffness matrix. This can be done by approximating the derivative with a finite differences operator. Let D_ε be the usual second-order finite differences approximation of d/dx, i.e., $D_\varepsilon\phi(x) \equiv \phi(x+\varepsilon) - \phi(x-\varepsilon)$, we define an approximation of d/dx, denoted by $\partial_h^{[2n]}$, as a linear combination of the form $\partial_h^{[2n]} = \dfrac{1}{h}\displaystyle\sum_{p=1}^{n} \nu_p D_{(2p-1)h/2}$ which is of order $2n$ provided that the coefficients satisfy the Van der Monde like system

$$\sum_{p=1}^{n} \nu_p (2p-1)^{2k-1} = \delta_{k1} \quad \text{for } 1 \le k \le n.$$

This leads to the stiffness matrix $K^{[2n]}_{ij} \equiv (c\partial_h^{[2n]}\chi_i, \partial_h^{[2n]}\chi_j) \equiv (k^{[2n]}\chi_i,\chi_j)$ (here $k^{[2n]}$ is the notation for the operator $(\partial_h^{[2n]})^\star(c\partial_h^{[2n]})$). The $2n$-order **classical scheme** uses these two matrices and thus leads to the solution of the linear system

$$\begin{cases} \dfrac{dw_h}{dz} - i\omega M^\star \varphi_h = 0 \\ \left(\omega^2 M^\star - aK^{[2n]}\right)\varphi_h = bK^{[2n]} w_h, \end{cases}$$

We propose here another family of schemes, **the modified schemes**, that can be seen as an extension of Claerbout's scheme [6] to higher orders. It consists in using a linear combination $M^{[2n]}_\alpha = \alpha M^\star + (1-\alpha)U^{[2n]}$, of two different mass matrices $M^\star$ and $U^{[2n]}$, where $U^{[2n]}$ comes from a $2n$-order finite differences approximation of the operator identity (see below). The linear system associated to the **modified schemes** can then be written as

$$\begin{cases} \dfrac{dw_h}{dz} - i\omega M^\star \varphi_h = 0 \\ \left(\omega^2 M^{[2n]}_\alpha - aK^{[2n]}\right)\varphi_h = bK^{[2n]} w_h, \end{cases} \tag{9}$$

It is important to keep the diagonal mass matrix $M^\star$ in the first equation as we will have to invert it when we eliminate φ_h. On the other hand, one will show that a proper choice of α allows us to gain two orders of accuracy when compared to the classical discretization. The classical schemes correspond to the particular choice $\alpha = 1$ in (9).

Let us explain in more details how we derive $U^{[2n]}$. We use the same kind of procedure than for the stiffness matrix, and set I_ε the finite difference operator approximating the identity, i.e., $I_\varepsilon\phi(x) = \frac{1}{2}(\phi(x+\varepsilon)+\phi(x-\varepsilon))$. We define an approximation of I, denoted by $\delta_h^{[2n]}$, as a linear combination of the form $\delta_h^{[2n]} = \frac{1}{h}\sum_{p=1}^{n}\mu_p I_{(2p-1)h/2}$ which is of order $2n$ provided that the coefficients satisfy

$$\sum_{p=1}^{n}\mu_p(2p-1)^{2(k-1)} = \delta_{k1} \quad \text{for } 1 \le k \le n.$$

There is a very simple relation between the coefficients ν_p for the approximations of $\partial_h^{[2n]}$ and μ_p for the approximations of $\delta_h^{[2n]}$

$$\mu_p = (2p-1)\nu_p \quad 1 \le p \le n. \tag{10}$$

This gives the approximate mass matrix $U_{ij}^{[2n]} = (\frac{1}{c}\delta_h^{[2n]}\chi_i, \delta_h^{[2n]}\chi_j)$, and the resulting combination $M_\alpha^{[2n]} = \alpha M^* + (1-\alpha)U^{[2n]}$ is still a $2n$-order approximation (again we denote by $m_\alpha^{[2n]}$ the associated operator equal to $\frac{\alpha}{c}I + (1-\alpha)(\delta_h^{[2n]})^*(\frac{1}{c}\delta_h^{[2n]})$). To determine the order of the resulting scheme (9), we consider a smooth solution (w,φ) of (8), and perform a Taylor expansion, to get

$$\left(\omega^2 m_\alpha^{[2n]} - ak^{[2n]}\right)\varphi - bk^{[2n]}w = \frac{\omega^2}{c}\varphi + c\frac{\partial^2}{\partial x^2}(a\varphi+bw)$$
$$+h^{2n}\frac{\partial^{2n}}{\partial x^{2n}}\left[\frac{\omega^2}{c}(1-\alpha)C_1^{[2n]}\varphi + cC_2^{[2n]}\frac{\partial^2}{\partial x^2}(a\varphi+bw)\right] + O(h^{2n+2}),$$

where $C_1^{[2n]}$ and $C_2^{[2n]}$ are constants computed thanks to Taylor expansions with respect to h (we omit the details). The first term in the RHS is exactly the equation satisfied by (w,φ) and thus vanishes. The order of the scheme is therefore at least equal to $2n$. Moreover, one again recognizes the same equation between the brackets for an appropriate choice of α which is $\alpha = \alpha^{[2n]} = 1 - \frac{C_2^{[2n]}}{C_1^{[2n]}}$ and hence we obtain a $2n+2$-order scheme with that particular choice for α. Finally, relation (10) provides an explicit expression for this value, $\alpha^{[2n]} = \frac{2n}{2n+1}$. From a practical point of view, relation (10) is very useful, since it is sufficient to compute a $2n$-order approximation of d/dx to get at the same time the $2n$-order approximation of the identity.

With $n=1$, the modified scheme corresponds to the classical Claerbout's scheme [6], with the relation $\gamma = (1-\alpha)/4$. The corresponding value $\alpha^{[2]} = 2/3$ leads then to the well known $\gamma = 1/12$ choice and gives a fourth-order scheme with a matrix of bandwidth equal to 3 instead of 7 for the classical fourth-order scheme. For $n=2$ and $\alpha = 4/5$, we get a sixth-order one with a bandwidth equal to 7 instead of 11 for the classical sixth-order scheme. The drawback is that the analysis through a priory energy estimates, that can be made for the classical schemes to show their well-posedness, is not valid anymore for these modified schemes in heterogeneous media (although it still applies to homogeneous media)(see Collino [7]).

Before closing this section, let us write the total discretization. System (9) can be rewritten

after elimination of the auxiliary function as

$$\frac{dw_h}{dz} = i\omega M^*(\omega^2 M_\alpha^{[2n]} - aK^{[2n]})^{-1} bK^{[2n]} w_h. \tag{11}$$

The discretization in depth is introduced as in section 3, and we finally end up with a set of linear systems of the form

$$\bar{S}_k w_{k+1} = S_k w_k \ ,$$

with $S_k = (\frac{\omega^2}{a} M_\alpha^{[2n]} - K^{[2n]})(M^*)^{-1} + r_k \frac{\omega \Delta z b}{a} K^{[2n]}$ and $\bar{S}_k$ its complex conjugate.

5 Numerical results

We will illustrate the method with two examples. For both examples, we consider the migration of a point source, the source being located on the surface. The computational domain is 1250 m in each of the horizontal directions, and 625 m in the vertical direction. The grid sizes are $h = \Delta z = 12.5$ m. The first simulation is done in a 2D heterogeneous medium (see figure 1), the

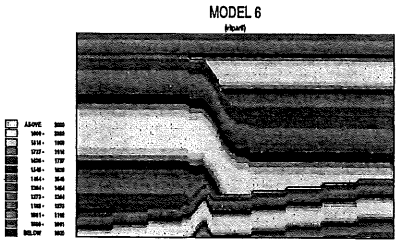
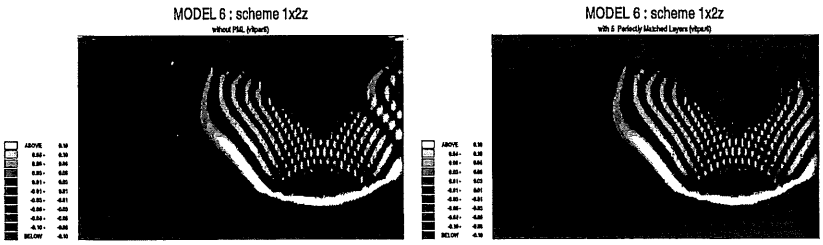


Figure 1: **Smooth heterogeneous medium**

central frequency of the source is around 28 Hz, and the cutoff frequency around 76 Hz.
The second simulation is done in a 3D homogeneous medium with velocity equal to 1000m/s. We use the forty five degree paraxial equation with 4 directions and 1 fraction per direction, characterized by the coefficients $a_1 = a_4 = 0.27$, $b_1 = b_4 = 0.3$ in the directions x_1 and x_2 and $a_2 = a_3 = 0.41$, $b_2 = b_3 = 0.2$ in the diagonal directions. The central frequency of the source is around 20 Hz, and the cutoff frequency around 50 Hz. Both tests are relatively severe since the number of points per wavelength at the cutoff frequency is around 1.5.

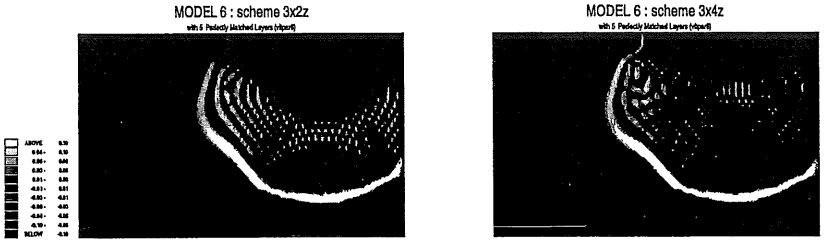
As announced in section 2.3, figures 2 show that the reflexions on the lateral boundary due to Dirichlet boundary conditions have almost completely disappeared using PMLs of size equal to $5h$ (notice that this experiment corresponds to the heterogeneous case).

As explained in section 4, the main drawback of modified schemes compared to the classical ones is to achieve the same accuracy for a lower price (lower bandwidth linear systems). Moreover, the numerical experiments show their superiority from a dispersion point of view as can be seen on figures 5,6 and this is confirmed by a dispersion analysis (see Bécache et al [3], for further numerical



(a) **DIRICHLET BC** (b) **5 PML**

Figure 2: **Classical second-order scheme**

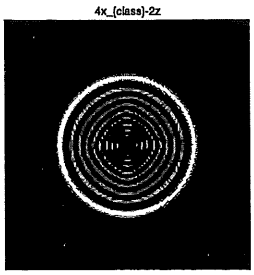


(a) 5 PML ; **2nd-order** in z (b) 5 PML ; **4th-order** in z

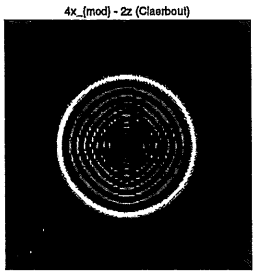
Figure 3: **Modified 4th-order schemes in** x

(a) 5 PML ; **2nd-order** scheme in z (b) 5 PML ; **4th-order** scheme in z

Figure 4: **Modified 6th-order schemes in** x

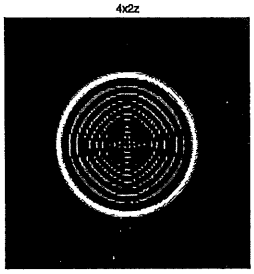


(a) **Classical** 4th-order in x and 2nd-order in z

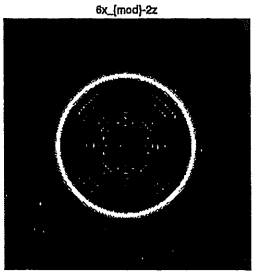


(b) **Modified** 4th-order in x and 2nd-order in z

Figure 5: **Classical and modified 4th-order schemes (2nd-order in z)**

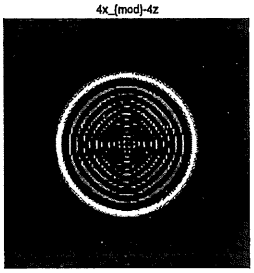


(a) **Classical** 6th-order in x and 2nd-order in z

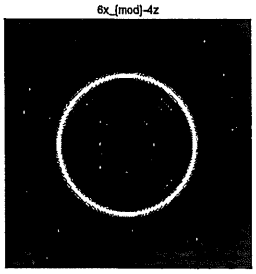


(b) **Modified** 6th-order in x and 2nd-order in z

Figure 6: **Classical and modified 6th-order schemes (2nd-order in z)**



(a) **Modified 4th-order** scheme (4th-order in z)



(b) **Modified 6th-order** scheme (4th-order in z)

Figure 7: **Fourth-order schemes in z**

experiments and for the dispersion analysis).

The simulations in the heterogeneous 2D medium, and for $\Delta x = \Delta z$, show a still good improvement when we use a modified 4th-order discretization in x instead of a 2nd-order one (compare Figures 2(b) and 3(a)) and even better when we also use a 4th-order discretization in z (see Figure 3(b)). Nevertheless, the improvement is not clear when we increase the discretization order in x until the order 6 (see Figure 4).

In the second example (3D homogeneous), again the gain is really important by using a 4th-order discretization in the lateral variables instead of a 2nd-order one. But this time the improvement using a 4th-order discretization in z is not obvious (compare figures 5(b) and 7(a) for the 4th-order modified scheme and figures 6(b) and 7(a) for the 6th-order modified scheme) although the dispersion is much more attenuated with the modified 6th-order schemes (see figure 6(b) for the 2nd-order discretization in z). Also one should notice in the homogeneous case the quite good isotropy obtained with these new paraxial equations despite of the introduction of particular directions used for the splitting.

These two examples lead to different conclusions concerning the optimal choice for the scheme. In order to better understand the behavior of the different schemes, we analyze in [3] their dispersion relations. This helps to choose the scheme with respect to the frequency, the number of points per wavelength... Of course, since we want the result in the time domain, we have to handle with a quite large number of frequencies and the optimal choice is not the same for all of them. A possible way for further improvements could be to adapt the choice of the numerical scheme to the frequency.

6 CONCLUSION

We have presented a way to solve efficiently and accurately new 3D paraxial equations that lend themselves to splitting with respect to the lateral variables in a consistent manner. This gives rise to the solution of a sequence of 1D linear systems for each splitting direction. This is done in a rather general context that handles heterogeneous media as well as the treatment of the lateral boundaries. The numerical dispersion occuring with the classical second-order schemes can be considerably attenuated with the use of the modified higher order schemes, even with coarse discretization grids.

One should also compare the efficiency of these equations with the full paraxial equation. This is work in progress and will be the subject of a forthcoming paper.

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