Theoretical and Computational Acoustics
D. Lee, Y-H Pao, M.H. Schultz, and Y-C Teng (Editors)

On the Use of Stair Steps to Approximate Bathymetry Changes in Ocean Waveguides

Finn B. Jensen
SACLANT Undersea Research Centre
Viale San Bartolomeo 400, 19138 La Spezia, Italy

Abstract: Stair-step discretization criteria have been established for accurately representing smoothly varying bathymetry changes in numerical models. It is shown that the strictest criterion applies to backscatter calculations, where the horizontal stair-step size must be a small fraction of an acoustic wavelength ($\Delta r \leq \lambda/4$). The forward scatter problem — assuming that backscatter is weak — can be accurately solved with an order-of-magnitude larger step sizes. A coupled-mode approach is used to illustrate solution convergence by computing backscattering from a single bottom facet in a shallow-water waveguide. Published field solutions for the ASA benchmark wedge are used to illustrate forward scatter results.

1. Introduction

The use of small stair steps (Fig. 1) to represent smoothly varying bathymetry in ocean waveguides is common to most numerical solution techniques (parabolic equations, coupled modes, finite differences) employed today to solve range-dependent propagation and scattering problems [1]. It is assumed, of course, that the discrete-problem solution converges to the smooth-problem solution for an increasing number of stair steps. The issue to be addressed here is how many stair steps are required to obtain a smooth-problem solution. Physically one expects that the steps must be small (in some sense) compared to the acoustic wavelength, and we would like to determine how small, and whether the same criteria apply to forward and backscattered field calculations. The issue is important since the computational effort involved in computing a full two-way field solution increases quadratically with the number of stair steps used in discretizing the bathymetry variations.

2. Plane-Wave Scattering at Stair Steps

2.1 BACKSCATTERING

A simple model of the scattering process can be established by assuming that each stair step acts as a point scatterer, and that the full stair case consequently acts as an array of point scatterers. This is schematically illustrated in Fig. 2, where we consider a bottom facet with slope θ, approximated by a number of stair steps with horizontal spacing Δx. The incident plane wave is taken to be horizontally propagating.

It is well known that a sparse array of point sources can have several diffraction lobes corresponding to directions of phase coherent radiation by all elements of the array. Only if the array is dense enough, i.e. element spacing less than $\lambda/2$, is there only one diffraction lobe, which in this case would correspond to a specular reflection of the incident plane wave in the direction $\varphi = 180 - 2\theta$. For large stair steps and hence large element spacing, there are several directions in which the scatters radiate in phase.

With reference to Fig. 2, it is easy, from simple geometric considerations, to determine the directions φ_n for all possible diffraction lobes. We just write down the conditions for the path length difference $\Delta x + d$ being an integer number of wavelengths, that is

$$\cos(\varphi_n + \theta) = \cos\theta \left[\frac{n\lambda}{\Delta x} - 1 \right], \qquad n = 0, 1, 2, ... \tag{1}$$

It is clear that if we have several diffraction lobes contributing to the backscattered field, the solution is wrong. Only the fundamental lobe $n = 0$ has physical meaning for a smooth bottom facet. The criterion for having only one diffraction lobe is found from Eq. (1) to be

$$\cos\theta \left[\frac{\lambda}{\Delta x} - 1 \right] > 1, \tag{2}$$

or

$$\Delta x < \lambda \frac{\cos\theta}{1 + \cos\theta}. \tag{3}$$

For small slope angles, this criterion is seen to be equivalent to the $\lambda/2$ element spacing of a dense array. In practice, a slightly stricter criterion must be adopted to include steeper facet slopes. The numerical results presented in Sect. 3.1 indicate that $\Delta x \leq \lambda/4$ is an appropriate discretization criterion for facet slopes of up to 60°.

2.2 FORWARD SCATTERING

Plane-wave scattering from stair steps in the forward direction is illustrated in Fig. 3. The physical process is here quite different from the one outlined above for backscattering. Thus, forward scatter is associated mainly with reflections off the horizontal interfaces of length Δx, whereas contributions from the vertical steps of height Δy can be ignored.

The mean facet slope is again θ, and the incident plane wave is tilted downwards φ with respect to horizontal. Reconstructing the phase front for the reflected wave, we see that this wave is not 'plane' as it would have been, had it been reflected from a smooth facet. Instead, the wavefront is 'ragged' with sharp discontinuities corresponding to adjacent rays reflected from different stair steps. Note that the propagation direction corresponding to a 'mean' wavefront (dashed line in Fig. 3) forms an angle of $\varphi + 2\theta$ with the horizontal, as would a specularly reflected plane wave from a smooth bottom facet of slope θ.

The question is how distorted the wavefront can be before the specular reflection picture breaks down. It is reasonable to assume that the wavefront distortions must be small compared to a wavelength, or, from simple geometrical considerations,

$$\Delta y << \frac{\lambda}{2\sin\varphi} \ . \tag{4}$$

This criterion can be written also in terms of Δx as

$$\Delta x << \frac{\lambda}{2\sin\varphi\,\tan\theta} \ . \tag{5}$$

It is easily seen that allowable step sizes for forward scatter are several wavelengths for the standard small-slope problems encountered in ocean acoustics. In Sect. 3.2 we quantify the above discretization criterion for forward propagation in the ASA benchmark wedge, and compare to published numerical results.

3. Numerical Results

The acoustics literature reporting numerical solutions for forward scatter in range-dependent ocean waveguides is abundant, and it is not difficult to find examples that illustrate the effects of coarse stair-step approximations on solution accuracy. However, when dealing with backscatter the situation is quite different. A concerted effort to generate accurate numerical solutions for backscattering in ocean waveguides due to bottom features of different shapes and heights was initiated just recently [2,3]. As of today few published numerical solutions are available for checking the effect of stair-step discretization on the accuracy of computed backscatter in ocean waveguides.

To generate accurate numerical solutions for backscatter in a Pekeris waveguide with a single bottom facet (Fig. 1) we employ a coupled-mode code (COUPLE) developed by Evans some years ago [4]. This code was recently updated to include a 'sponge' layer deep in the bottom [5], thus improving the computational performance by a factor 20–50. COUPLE was successfully applied to a series of benchmark problems involving backscattering [2], and is considered a reference code for this type of work.

3.1 BACKSCATTERING

We consider the test problem shown in Fig. 1. The environment consists of a 200-m deep shallow-water waveguide bounded above by a pressure-release surface and below by a penetrable, homogeneous fluid bottom. The water column is isovelocity with $c_W = 1500\,\mathrm{m/s}$. The bottom properties are $c_B = 1700\,\mathrm{m/s}$, $\alpha_B = 0.5\,\mathrm{dB}/\lambda$, and $\rho_B = 1.5\,\mathrm{g/cm^3}$. We consider a 2D problem with translational symmetry in the y-direction. The obstacle is a 35-m $(7\,\lambda)$ high protrusion on the bottom placed 1.5 km downrange and having the same acoustic properties as the seabed. Thus there is only one scattering facet, namely the front-end of the protrusion. The source is a 300-Hz Gaussian beam directed towards the scattering facet. As shown in Fig. 4, this beam provides a uniform insonification of the front-end of the obstacle.

For a facet slope of 30° we shall investigate the effect of the stair-step discretization on the computed backscatter. An accurate total field solution is displayed in Fig. 4, and it is clear that energy is scattered primarily in the forward direction. This reference solution is done with 150 stair steps across the facet, i.e.

$\Delta x = 7\lambda/(150 \cdot \tan 30°) \simeq \lambda/12$. Hence, this is an accurate sampling compared to the earlier derived discretization criterion of $\Delta x \leq \lambda/4$. Additional results for scattering from a single bottom facet may be found in a companion paper [3] addressing the effect of height and slope of the facet on the strength of the backscattered field.

We now turn to a display of just the backscattered field computed for the 30°-facet discretized by an increasing number of stair steps (NSS). The first set of results are given in Fig. 5 for NSS = 10, 11, 12 and 14. Thus, in the upper display there are 10 stair steps, and we see two backscattered beams marked n=1 and n=2. If we compute the angles associated with the diffraction lobes for this case as given by Eq. (1), we find $\varphi_0 = 120°$ (the specularly reflected beam going forward and not seen is the display), $\varphi_1 = 68.6°$, and $\varphi_2 = 25.6°$. The angles computed from our simplified scattering model are in excellent agreement with those determined form the full field display. By increasing the number of stair steps the two diffraction beams become more horizontal, and for NSS = 12 the 2nd-order beam shoots straight back at the source. In the lower graph with NSS = 14, the 2nd-order beam has disappeared (passes though lower endfire of the scattering array), and we are left with only the 1st-order beam. Since this beam is steeper (52.3°) than the critical angle at the bottom (28.1°), little energy is propagated back to the source.

The next series of field plots for NSS = 17, 20, 24 and 30 are shown in Fig. 6. Now it is the 1st-order beam that moves to smaller angles and provide strong backscattering for NSS = 24. Finally, when the number of stair steps is 30, also the 1st-order diffraction lobe disappears, and we are left with only the zeroth-order beam, which is specularly reflected in the forward direction. The very low backscatter levels seen in the lower panel is the only result here that resembles the correct solution for a smooth facet.

By computing the mean intensity over depth (0–200 m) of the backscattered field at the source range, we can summarize the convergence process for increasing number of stair steps in a single graph, see Fig. 7. Note that we generally have high backscatter levels ($\sim$ 45 dB) for $\Delta x > \lambda/2$ followed by a rapid transition to much lower levels ($\sim$ 90 dB) for $\Delta x < \lambda/4$. As explained earlier the high backscatter levels are caused by coherent backscatter into higher-order diffraction lobes associated with a sparse array of point scatterers. The oscillatory pattern is a critical-angle effect, and low backscatter levels are seen for instance for NSS = 14 (Fig. 5). Clearly, the high backscatter levels are the result of using too few stair steps for representing the smooth bottom facet. The appropriate discretization criterion is $\Delta x \leq \lambda/4$, with a smoothly converging answer for an increasing number of stair steps. Often this criterion will provide satisfactory solution accuracy, but for benchmarking purposes even smaller steps may be required, $\Delta x = \lambda/10$ to $\lambda/20$.

As s further example of solution convergence, we show in Fig. 8 the computed mean backscatter level for a facet slope of 60°. In this case the specularly reflected beam is in the backward direction ($\varphi_0 = 60°$). The facet hight is again 7λ, but since the horizontal extent of the facet is shorter, fewer stair steps are required for an accurate solution. The transition from stair-step scattering to smooth-facet scattering is again seen to occur rapidly, with a level drop of around 35 dB.

The standard approach to ensuring accurate numerical results is by increasing the number of sample points (stair steps) until the solution converges. In the present case, such an approach could lead to a completely wrong result. Thus computations with NSS = 2, 4 and 8 would give almost identical answers, but it is the solution to a 'rough'-facet problem of some kind. The answer to the posed scattering problem involving a smooth facet is obtained by using $\Delta x \leq \lambda/4$.

3.2 FORWARD SCATTERING

If the general scattering problem due to changing bathymetry involves strong backscatter, then the full two-way field solution will require stair-step discretization according to the earlier sampling criterion, i.e. $\Delta x \leq \lambda/4$. On the other hand, if we are dealing with small-slope problems where backscatter is weak, then the forward solution can be accurately computed in a single-scatter approach (backscatter neglected), and much larger stair steps are generally permissible. In Sect. 2 we derived a discretization criterion, Eq. (5), which shall be tested on the ASA wedge benchmark [6].

This test problem involves a sloping-bottom environment with the same acoustic parameters as given in Fig. 1. The water depth changes from 200 m at the source to 0 m at a range of 4 km. The bottom slope is 2.86°, and the outgoing field solution is sought for a frequency of 25 Hz ($\lambda = 60$ m). The original benchmark solution was generated with NSS = 200, or $\Delta x = \lambda/3$ [6]. Subsequently Collins [7] showed that very accurate forward solutions to this problem could be obtained with just 40 stair steps, or $\Delta x = 1.67\,\lambda$. Even the double step size of $\Delta x = 3.33\,\lambda$ gave fairly good results.

Let us see if our discretization criteria for forward scattering, Eq. (5), can be reconciled with these numerical results. First of all, the angle φ of the incident field must be determined. Since the step length decreases with increasing angle φ, a conservative estimate is obtained by using for φ the highest propagation angle of importance in the problem, i.e. the critical angle of 28.1°. We find from Eq. (5) that the step size Δx must satisfy the criterion $\Delta x << 20\,\lambda$. Assuming that it is sufficient to decrease the upper limit by an order-of-magnitude we obtain $\Delta x \leq 2\,\lambda$, which is in agreement with the numerical results of [6,7]. Note that forward-scatter problems generally can be done with much larger stair steps than backscatter problems, which means that forward problems are computationally easier, often by 1–2 orders of magnitude in CPU time.

4. Conclusions

By using a simple plane-wave model of scattering from a series of stair steps approximating a smooth bottom slope, we established discretization criteria for accurate numerical solution of the facet scattering problem. The strictest criterion was found to apply to backscatter calculations where the horizontal step size must be smaller than $\lambda/4$ to guarantee accurate numerical results. Actually, a completely different stair-step scattering problem is solved if the step size is larger than $\lambda/2$. The transition from stair-step scattering to facet scattering occurs rapidly for $\lambda/2 > \Delta x > \lambda/4$. For small bottom slopes where backscatter is weak, the forward problem can be solved with much larger horizontal steps, often of the order of several wavelengths.

References

[1] F.B. Jensen, W.A. Kuperman, M.B. Porter, and H. Schmidt, *Computational Ocean Acoustics* (Amer. Inst. of Physics, New York, 1994).

[2] S.A. Chin-Bing, D.B. King, J.A. Davis and R.B. Evans, eds., *R&S Workshop: Proceedings of the Reverberation and Scattering Workshop* (Naval Research Laboratory, Stennis Space Center, MS, 1995).

[3] F.B. Jensen, "Backscatter calculations in simple waveguide geometries using stepwise coupled normal modes," (these proceedings)

[4] R.B. Evans, "A coupled mode solution for acoustic propagation in a waveguide with stepwise depth variations of a penetrable bottom," J. Acoust. Soc. Amer. **74**, 188–195 (1983).

[5] R.B. Evans, "A reverberation calculation using stepwise coupled modes," (in Ref. [2]).

[6] F.B. Jensen and C.M. Ferla, "Numerical solutions of range-dependent benchmark problems in ocean acoustics," J. Acoust. Soc. Amer. **87**, 1499–1510 (1990).

[7] M.D. Collins, "A split-step Padé solution for parabolic equation methods," J. Acoust. Soc. Amer. **93**, 1736–1742 (1993).

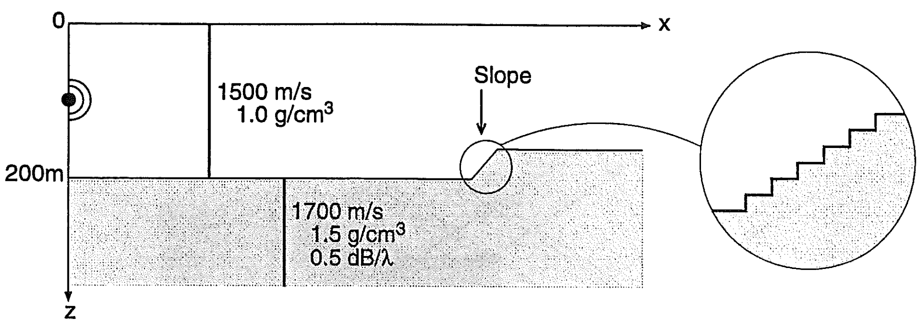

Figure 1: Schematic of Pekeris waveguide with a single-facet protrusion on the seafloor.

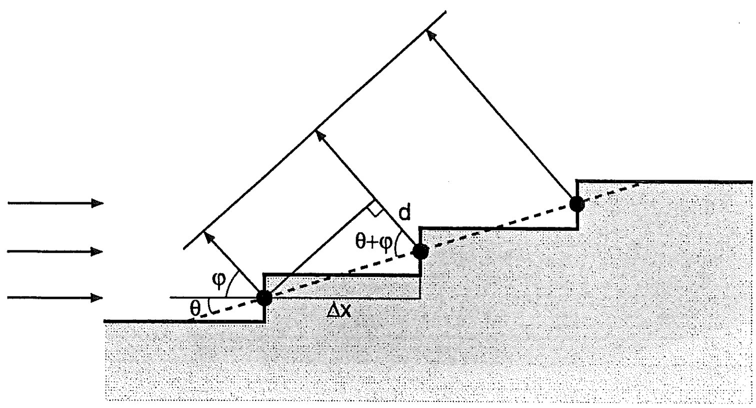

Figure 2: Geometry for computing coherent backscatter from individual stair steps.

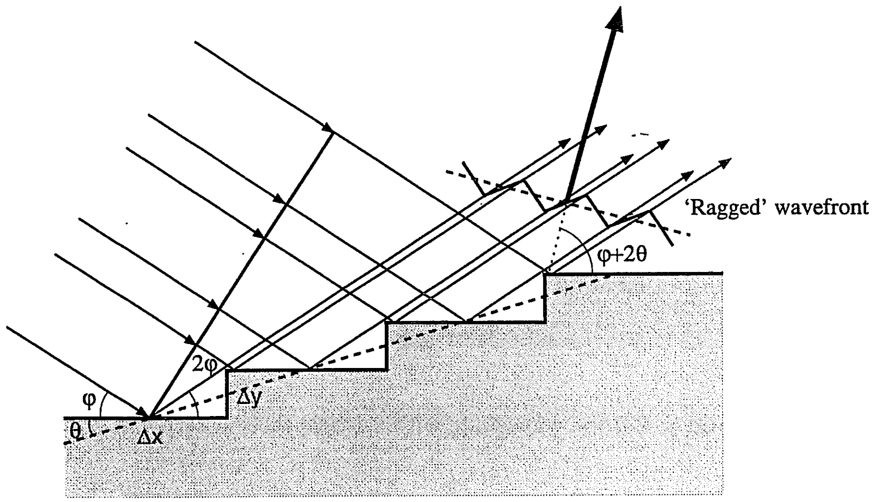

Figure 3: Geometry for computing coherent forward scatter from individual stair steps.

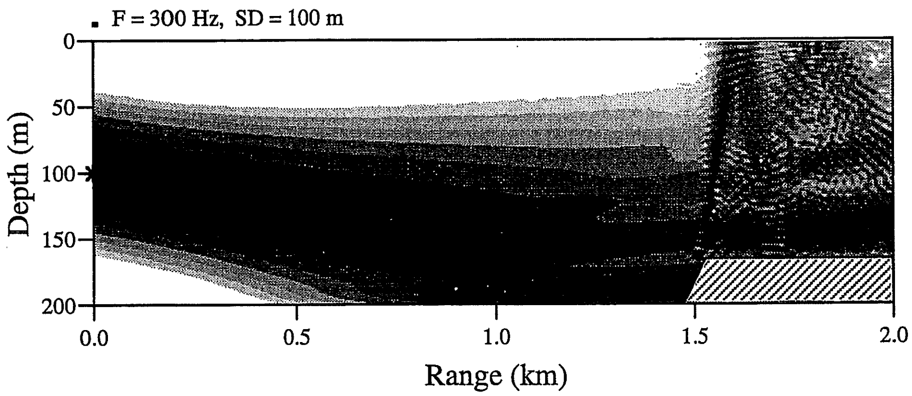

Figure 4: Beam insonification of obstacle placed on the bottom 1.5 km from the source. The contour levels (from black to white) are 30 to 60 dB in steps of 5 dB.

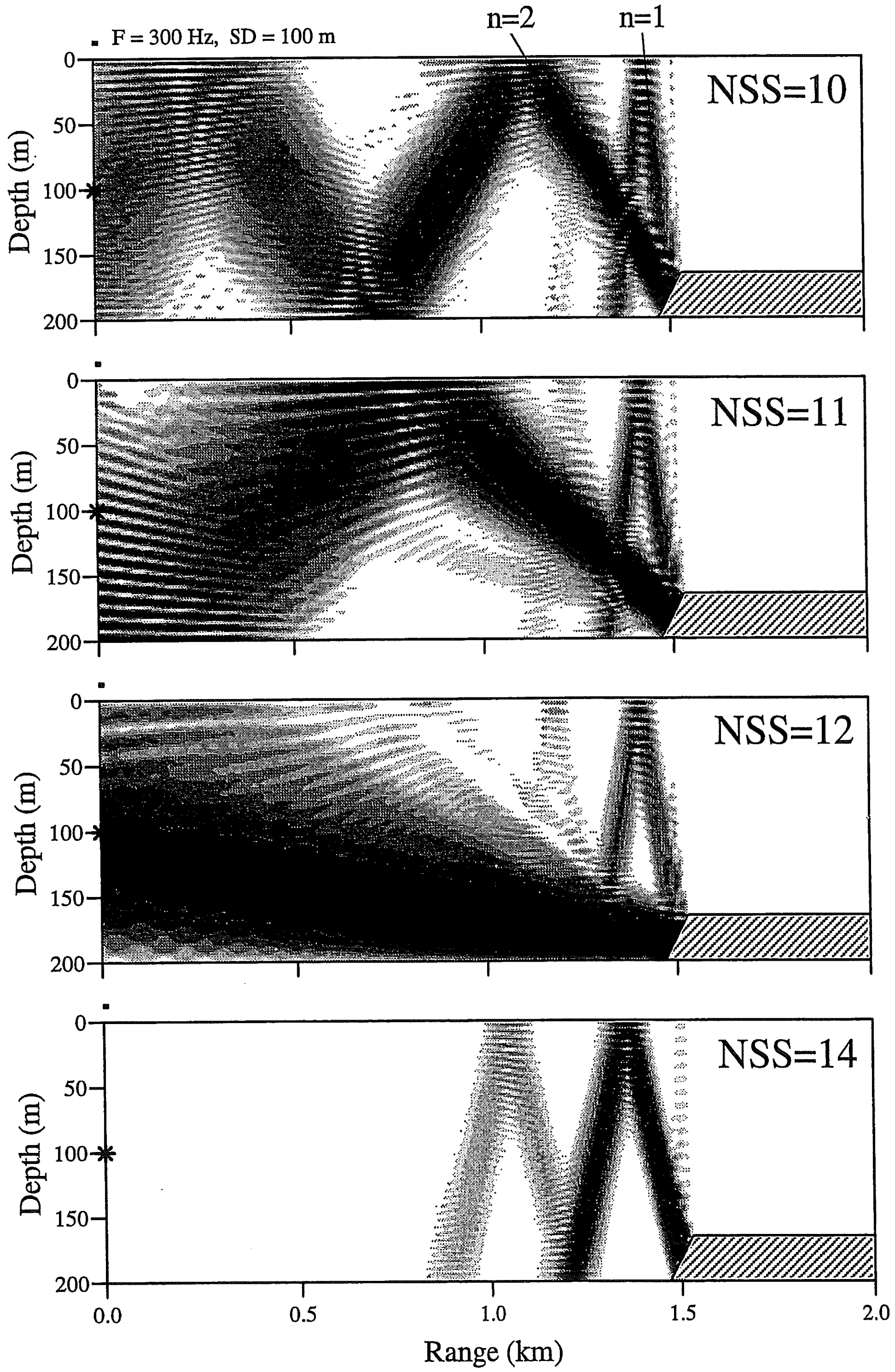

Figure 5: Backscattered field for different number of stair steps (NSS) across the 30° bottom facet. The contour levels (from black to white) are 40 to 61 dB in steps of 3 dB.

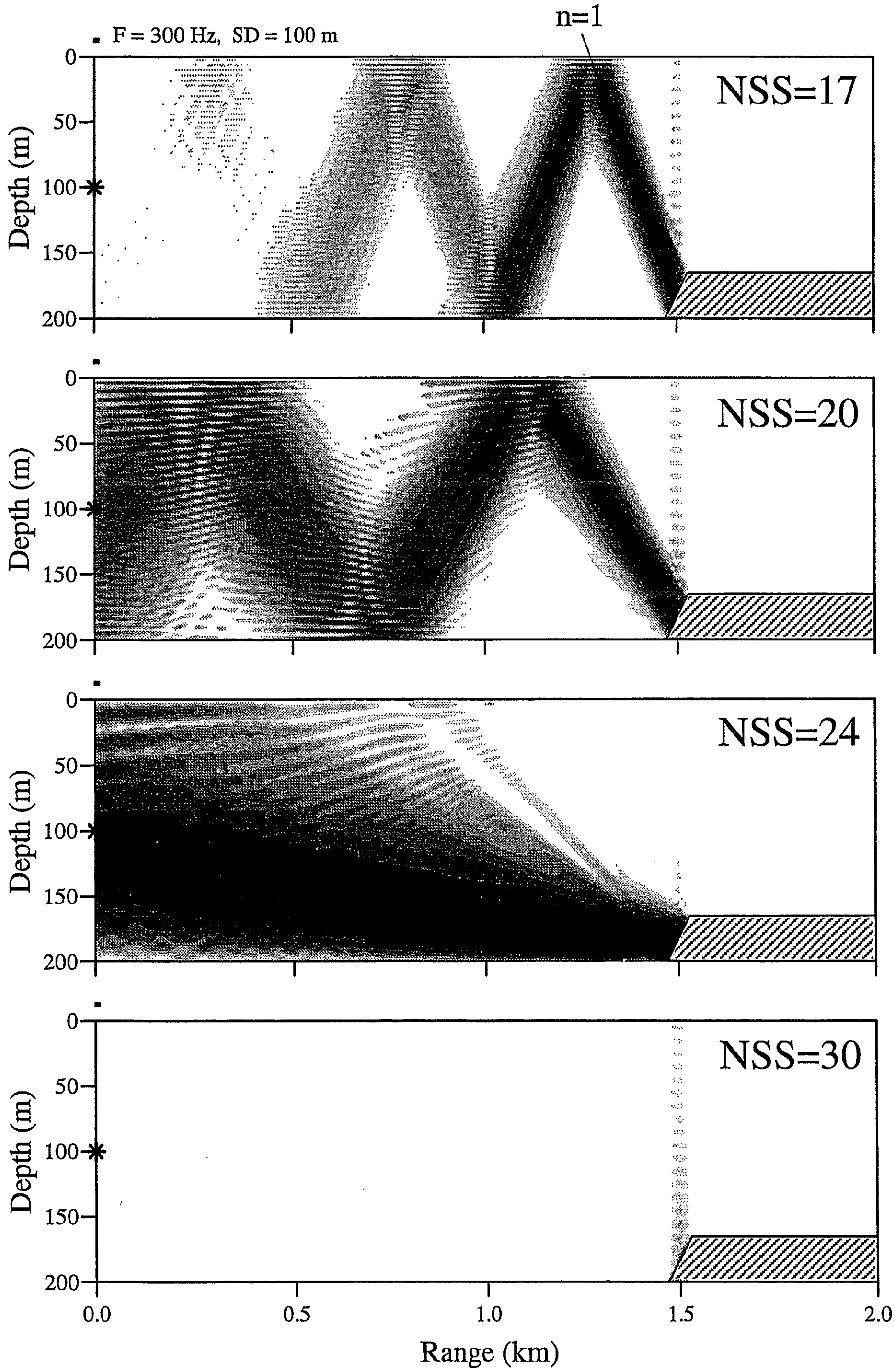

Figure 6: Backscattered field for different number of stair steps (NSS) across the 30° bottom facet. The contour levels (from black to white) are 40 to 61 dB in steps of 3 dB.

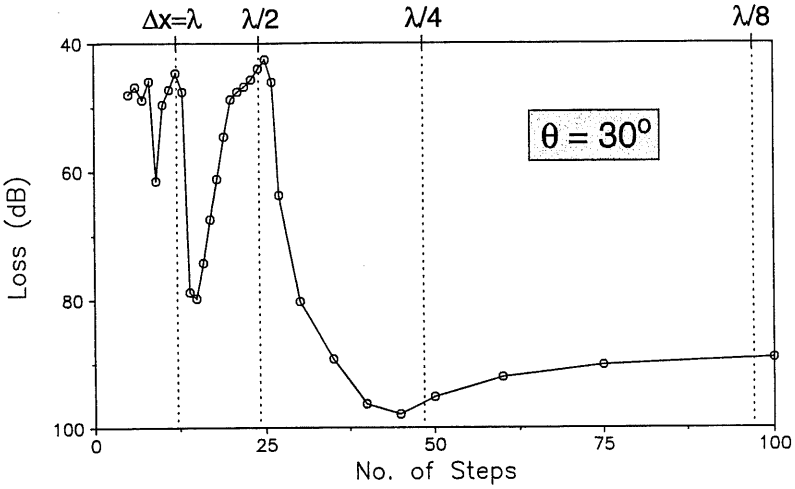

Figure 7: Mean backscatter level as a function of the number of stair steps used for approximating the sloping bottom facet ($\theta = 30°$).

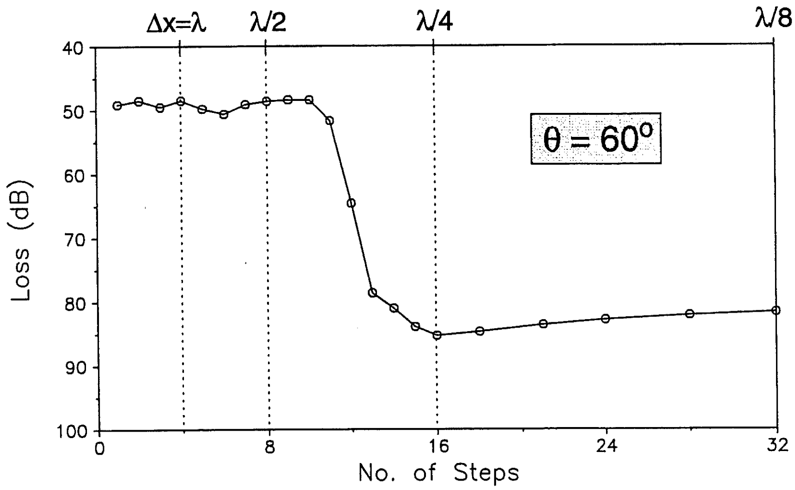

Figure 8: Mean backscatter level as a function of the number of stair steps used for approximating the sloping bottom facet ($\theta = 60°$).