THE ENERGY-CONSERVING PROPERTY OF THE STANDARD PE

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In 1974 a model was introduced by Frederick D. Tappert for predicting long-range wave propagation in a range-dependent environment. He applied the parabolic Equation approximation to transform the Helmholtz equation into a parabolic equation, the very first Parabolic Equation (PE). A pressure-release surface boundary is considered along with an artificial bottom boundary treatment. This paper proved that the Tappert model is energy-conserving.

1. Introduction

Over the past quarter century, the authors had continuous technical interactions with Frederick D. Tappert who, in 1974, introduced a model which is to apply the parabolic equation approximation to transform the Helmholtz equation into a parabolic equation, the very first parabolic equation (PE). In 1984 the first author invited Tappert to spend a summer together to do research in relation to PE developments.The author raised a question to Tappert: You made a big contribution of PE to the acoustic community; in order to honor your contribution, should the very first PE be named after you? Tappert said: No, but suggested naming it the Standard PE. From that time on the Standard PE was recognized by the acoustics community.

The Standard PE is a 2-dimensional (range and depth) representative wave equation which defines an initial-boundary value problem. Associated with the Standard PE, the surface boundary condition is considered pressure-release; the bottom boundary condition is treated by a special technique, introduced by Tappert, called "artifical bottom". The artifical bottom technique is to extend the field vertically down to the bottom deep enough such that $u(r, z) = 0$ at the bottom. The Standard PE to go with the assumed boundary conditions is regarded as the Tappert model.

Since the early 1980's, the authors and Tappert continued their technical discussions including the issues, contributions, and new results with reference to the PE-related developments. Various topics were among their discussions; Standard PE was one of the topics we discussed, but the energy-conserving issue entered the discussions but we did not pursue to prove that the Tappert model is energy-conserving. This paper is to prove that the Tappert model is energy conserving.

2. Basic Development

This section consists of the outline of two parts: The theoretical development of the standard PE and the associated surface and bottom boundary conditions. Theoretical details can be found in references [1, 2].

2.1. *The standard PE*

Let r be the range variable, z be the depth variable, $u(r, z)$ the 2-dimensional wave field, $n(r, z)$ be the index of refraction which is a real-valued function, and k_0 is the reference wavenumber which is a real-valued scalar.

The very first Parabolic Equation introduced by Tappert [1] takes the form:

$$u_r = \frac{ik_0}{2}(n^2(r, z) - 1)u + \frac{1}{2ik_0}\frac{\partial^2 u}{\partial z^2}\,. \tag{2.1}$$

2.2. *Associated boundary conditions*

Two types of boundary conditions are considered: the surface and the bottom boundaries. Let z_s indicate the surface boundary and z_b, the bottom boundary.

2.2.1. *The surface boundary condition*

The assumed pressure-release surface condition indicates that $u(r, z_s) = 0$. The indication implies that the prescribed surface boundary conditions are

$$u(r.z_s) = 0, \qquad \overline{u}(r, z_s) = 0, \qquad \frac{\partial u}{\partial z}|_{z_s} = 0, \qquad \frac{\partial \overline{u}}{\partial z}|_{z_s} = 0\,. \tag{2.2}$$

2.2.2. *The bottom boundary condition*

A technique was introduced by Tappert to generate the bottom boundary condition. This technique is known as the "artificial bottom" which is to extend the wave field vertically deep enough such that $u(r, z_b) = 0$ there. Therefore; the prescribed bottom boundary conditions are

$$u(r, z_b) = 0, \qquad \overline{u}(r, z_b) = 0, \qquad \frac{\partial u}{\partial z}|_{z_b} = 0, \qquad \frac{\partial \overline{u}}{\partial z}|_{z_b} = 0\,. \tag{2.3}$$

3. Energy-Conserving Property

Writing Eq. (2.1) in the form

$$u_r = a(n^2(r,z) - 1)u + bu_{zz} \tag{3.1}$$

where

$$a = \frac{ik_0}{2}, \qquad b = \frac{1}{2ik_0}. \tag{3.2}$$

From Eq. (3.1),

$$\overline{u_r} = \overline{a}(n^2(r.z) - 1)\overline{u} + \overline{b}\frac{\partial^2 \overline{u}}{\partial z^2}, \tag{3.3}$$

we have

$$u_r\overline{u} = a(n^2(r,z) - 1)u\overline{u} + b\left(\frac{\partial^2 u}{\partial z^2}\right)\overline{u}, \tag{3.4}$$

and

$$\overline{u_r}u = \overline{a}(n^2(r.z) - 1)\overline{u}u + \overline{b}\left(\frac{\partial^2 \overline{u}}{\partial^2 z}\right)u. \tag{3.5}$$

Then

$$u_r\overline{u} + \overline{u_r}u = b\left[\frac{\partial^2 u}{\partial z^2}\overline{u} - \frac{\partial^2 \overline{u}}{\partial^2}u\right] = \frac{\partial}{\partial r}(u\overline{u}) = \frac{\partial}{\partial r}(|u|^2). \tag{3.6}$$

We want to examine whether or not

$$\frac{d}{dr}\int_{z_s}^{z_b}|u|^2 dz = 0. \tag{3.7}$$

Making use of expressions in Eq. (3.6), we have

$$\frac{\partial}{\partial r}\int_{z_s}^{z_b}|u|^2 dz = \frac{\partial}{\partial r}\int_{z_s}^{z_b} b\left[\left(\frac{\partial^2 u}{\partial z^2}\right)\overline{u} - \left(\frac{\partial^2 \overline{u}}{\partial z^2}\right)u\right]dz. \tag{3.8}$$

Then, saving of the writing of $\frac{d}{dr}$ and the constant b, the first integral of the right-hand-side of Eq. (3.8) can be evaluated by means of integration by parts; i.e.

$$\int_{z_s}^{z_b}\left(\frac{\partial^2 u}{\partial z^2}\right)\overline{u}dz = \left(\frac{\partial u}{\partial z}\right)\overline{u}|_{z_b} - \left(\frac{\partial u}{\partial z}\right)\overline{u}|_{z_s} - \int_{z_s}^{z_b}\frac{\partial u}{\partial z}\frac{\partial \overline{u}}{\partial z}dz. \tag{3.9}$$

Similarly, the second integral of the right-hand-side of Eq. (3.8) becomes

$$-\int_{z_s}^{z_b}\left(\frac{\partial^2 \overline{u}}{\partial z^2}\right)udz = -\left(\frac{\partial \overline{u}}{\partial z}\right)u|_{z_b} + \left(\frac{\partial \overline{u}}{\partial z}\right)u|_{z_s} + \int_{z_s}^{z_b}\frac{\partial u}{\partial z}\frac{\partial \overline{u}}{\partial z}dz. \tag{3.10}$$

The term $(\frac{\partial u}{\partial z})\overline{u}|_{z_s}$ in Eq. (3.9) and the term $(\frac{\partial \overline{u}}{\partial z})u|_{z_s}$ in Eq. (3.10) all go to zero due to the surface boundary condition, (2.2).

Similarly, the term $(\frac{\partial u}{\partial z})\overline{u}|_{z_b}$ in Eq. (3.9) and the term $(\frac{\partial \overline{u}}{\partial z})u|_{z_b}$ in Eq. (3.10) all go to zero due to the bottom boundary condition, (2.3), therefore;

$$\frac{d}{dr}\int_{z_s}^{z_b}|u|^2dz = \frac{d}{dr}b\left[\int_{z_s}^{z_b}\left(-\frac{\partial u}{\partial z}\frac{\partial \overline{u}}{\partial z}\right)dz + \int_{z_s}^{z_b}\frac{\partial \overline{u}}{\partial z}\frac{\partial u}{\partial z}dz\right] = 0\,. \tag{3.11}$$

Then, the energy-conserving property of the standard PE with the prescribed boundary conditions, (2.2) and (2.3), is proved.

4. Remarks

The Standard PE is a two-dimensional model with a narrow angle capability. These days the three-dimensional models have become more realistic in real applications. Not many users in the scientific community are using the two-dimensional model. Why bother to study the energy-conserving property for the Standard PE?

There are a few answers for this question:

1. Because of the interest in three-dimensional problems, the Standard PE may not be used often in the acoustics community, to report this theoretical result to the public, we believe, may still interest the readers.
2. We selected the Tappert model to show it is energy-conserving, on the other hand, is to remember the late Tappert for what he did for the scientific community.
3. The technique, we used to examine the energy-conserving property, can be used to examine the energy-property of other PE models.

5. Conclusion

The PE influence to the acoustic community is huge. All further-developed PE's are in use widely in the acoustifics community; they were all derived from the standard PE which benefited the scientific community a great deal. The Standard PE, even now-a-days is having limited use, it must not be forgotten; interestingly, the energy-conserving property of the Tappert model should not be unmentioned. This procedure may be applied to investigate the energy-conserving property for all other PE's, PE-like models, or other types of wave propagation models.

5.1. *Dedication*

The impact of the PE to the scientific community is huge. In recognition of the PE contribution to the acoustic community, we cannot forget the Standard PE; and Frederick D. Tappert must be remembered. This paper is written in memory of our long time colleague Frederick D. Tappert.

Acknowledgments

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References

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A Dedication to Professor Tappert

Professor Frederick D. Tappert, who introduced the parabolic equation approximation to the acoustic community, passed away in 2001.

Professor Michael I. Tarodakis and Dr. Finn B. Jensen organized a special memorial session for Prof. Tappert at the 6th International Conference on Theoretical and Computational Acoustics (ICTCA) at Hawaii, Honolulu, U.S.A. August 11-15, 2003.

Professor Tarodakis and Dr. Jensen further encouraged the session speakers to contribute their articles to be included in the Proceedings of Theoretical and Computational Acoustics 2003. Their efforts in organizing this memorial session is appreciated by all of us.

In January, 2000, I visited Prof. Tappert in Miami, Florida, U.S.A. He expressed interest in contributing a paper to the 6th ICTCA. At that time, I started writing a paper on Revolutionary Influence of the Parabolic Equation Approximation to honor him. I continued to make progress on this article. At that stage, it was an article but, I left room for expansion. After the shocking news regarding Prof. Tappert, I immediately started writing another article entitled The Energy-Conserving Property of the Standard PE and dedicated it in memory of Prof. Tappert. Suddenly I was diagnosed with age-related Macular Degeneration. I had difficulty reading and writing. I was forced to stop writing this article, which I had planned to submit at the 2003 Hawaii ICTCA in the memorial session for Prof. Tappert, organized jointly by Prof. Tarodakis and Dr. Jensen. I was unable to submit the Energy Conserving paper on time. I felt very guilty for not being able to present this paper. After the conference, I was determined to complete writing this article, if possible. Prof. Er-Chang Shang came to help. With his help, this article has been completed. I thank Prof. Shang for his help and thank the committee chairs for giving me the opportunity to present this article at the 2005 Hangzhou ICTCA.

Professor Frederick D. Tappert has gone; his PE contribution will be remembered. This article is dedicated in memory of my long-time colleague, Frederick D. Tappert.

Ding Lee

Fredrick D. Tappert

April 21, 1940 - January 9, 2001

Frederick D. Tappert was born April 21, 1940 in Philadelphia, Pennsylvania. His parents, the Reverend Dr. Theodore G. Tappert and Helen Louise Carson Tappert, raised their family of four children in the Lutheran Theological Seminary in Philadelphia, where the Reverend Dr. Tappert was a noted theologian. Fred showed an early penchant toward mathematics and science and attended Central High School in Philadelphia, which recognized outstanding young men in this area. From there he went on to study engineering at Penn State University, funded by the Ford Foundation, where he graduated with a B.S. in Engineering Science with honors in 1962. Fred went on to pursue his Ph.D. in theoretical physics from Princeton University with a full scholarship from the National Science Foundation. He earned his Ph.D. in 1967.

Upon graduation Dr. Tappert was hired to the Technical Staff at Bell Laboratories in Whippany, New Jersey from 1967 - 1974, where he worked on plasma physics and high altitude nuclear effects, UHF radar propagation, solitons in optical fiber, and ocean acoustic surveillance systems. He left Bell Labs and became a Senior Research Scientist at the Courant Institute, at New York University from 1974 - 1978, where he performed research on controlled fusion and nonlinear waves, as well as ocean acoustics. It was at the Courant Institute that Fred first realized the impact that he could have upon students and thus his future took on even more meaning as the great professor and advisor began to emerge in Fred.

Fred realized his potential as an educator and scientist when he left the Courant Institute and joined the faculty at the University of Miami's Rosenstiel School for Marine and Atmospheric Science in August 1978. At RSMAS he taught graduate courses in ocean acoustics, occasional undergraduate courses in physics, and supervised the research of more than 25 awardees of M.S. and Ph.D. degrees. In addition, Professor Tappert carried out a vigorous program of sponsored research in the areas of ocean acoustics, and wave propagation theory and numerical modeling.

Dr. Tappert was a major participant in the ONR-sponsored initiative on "Chaos and Predictability in Long Range Ocean Acoustics Propagation." In this research he applied a recently developed 4-D (three space dimensional plus time) full-wave fully range-dependent parabolic equation (PE) ocean acoustic model to determine the limits of predictability of sound propagation and scattering. Since Dr. Tappert's most cited research was the original development of the PE numerical model, and he was also one of the originators of the concept of "ray chaos" in ocean acoustic propagation, this was a natural evolution for his research. Previously, Professor Tappert was a major participant in the ONR-sponsored "Acoustic Reverberation Special Research Project," the goal of which was to gain a scientific understanding of long-range low-frequency ocean surface and bottom reverberation by comparing numerical model predictions to measured acoustic data, taking into account high resolution environmental data. In that research Professor Tappert developed a PE model of bistatic reverberation, the predictions of which compared favorably with measurements.

In addition to his university research, Dr. Tappert was a consultant to many organizations involved in applied projects related to wave propagation theory and numerical modeling. This includes the DANTES project, sponsored by DARPA, in which he developed a novel technique call Broadband Matched Field Processing (BMFP) that localizes sources of acoustic transient signals using a back-propagation method.

In October 2001, Fred was awarded the Superior Public Service Award from the Office of Naval Research. It was at this time that he was undergoing the rigors of chemotherapy in hopes that he would have more time in his fight against pancreatic cancer. This recognition brought tremendous joy to Fred. Unfortunately, he succumbed to the cancer only three months later on January 9, 2002. In November 2002, he was also posthumously awarded the Pioneers in Underwater Acoustics Award by the Acoustical Society of America. His wife, Sally, and two sons, Andrew and Peter, were present in Cancun, Mexico, to receive this award in his honor.

Sally Tappert