THE OPTIMUM SOURCE DEPTH DISTRIBUTION FOR REVERBERATION INVERSION IN A SHALLOW-WATER WAVEGUIDE

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Abstract

An approach of extracting the modal back-scattering matrix from the reverberation data in shallow-water is proposed recently (Shang, Gao and Tang, 2002). The kernel matrix of the inversion is constructed by the square of the modal function. The singularity of this matrix (or the stability of the inversion) is the crucial issue to be considered. In this paper, we discuss this issue analytically for a Pekeris waveguide with limited mode number *M*. The method that we used for singularity analysis is to calculate the maximum value of the determinant of this kernel matrix. We found that there is an optimum source depth distribution corresponding to the maximum value of the determinant of the kernel matrix. That means that by choosing the optimum source depth distribution we can get the most stable inversion. The conclusion is that under a quite tolerant condition the matrix is not singular.

1. Introduction

The inversion of reverberation data is very attractive because reverberation data is easy to obtain and a lot of environmental information can be retrieved. In [1], an approach of extracting the modal back-scattering matrix from the reverberation data is proposed, some numerical simulations are conducted in [2] and the inversion based on reverberation data is presented in [3]. In this paper, the stability of the inversion and the optimum source-depth distribution with an ideal waveguide is discussed. Up to the Born approximation the reverberation field can be expressed as [1] :

$$p^s(z_s,z;r_c)=(2\pi/k_0r_c)\sum_m^M\sum_n^M\phi_m(z_s)\phi_n(z)S_{mn}\int dv\cdot\eta(r)\exp\{i(k_m+k_n)r\} \qquad (1)$$

where $\phi_m(z)$ is the normalized mode function, k_m is the modal wave number, z_s is the source depth, z is the receiver depth, r_c is the center range of the scattering area, S_{mn} is the matrix which describes the mode coupling feature at the scattering element and $\eta(r)$ describes the random fluctuation of the scattering element which could be the interface roughness or the volume inhomogeneities.

By using the mode filter at the receiving array, we can get the j-th mode component:

$$p_j(z_s;r_c)=\int p^s(z_s,z;r_c)\phi_j(z)dz=(2\pi/k_0r_c)\sum_{m=1}^M\phi_m(z_s)S_{mj}\int dv\cdot\eta(r)\exp\{i(k_m+k_j)r\} \qquad (2)$$

By taking the in-coherent part as the averaged reverberation intensity, the j-th component is

$$I_j(z_s;r_c) = (2\pi / k_0 r_c)^2 A \sum_{m=1}^{M} \phi_m^2(z_s) \exp\{-2(\beta_m + \beta_j) r_c\} \Theta_{mj}^2 \tag{3}$$

Where

$$\Theta_{mj}^2 = S_{mj}^2 \sigma^2 K_{mj} \tag{4}$$

$$K_{mj} = (1/A) \iint dv_1 dv_2 N(r_1, r_2) \exp\{i(k_m + k_j) r_1 - i(k_m + k_j) r_2\} \tag{5}$$

where A is the insonified area, σ is the standard deviation of η, and $N(r_1,r_2)$ is the normalized correlation function of η.

In previous work, the inverting for back-scattering has been done based on some priori assumption of the scattering such as the Lambert's scattering , in this way the matrix inversion reduce to a parameter inversion. Or based on the assumption of separability like

$$\Theta_{mn}^2 = \Theta_m \Theta_n \tag{6}$$

and in this way the matrix inversion reduced to a vector inversion. In this paper, we consider the inversion of the original matrix Θ_{mn} from reverberation data represented by eq.(3) with an ideal waveguide.

2. The singularity analysis of the kernel matrix

In [1], the inversion approach has been proposed by changing the source depth. As we can see from eq.(3), that the kernel matrix is constructed by $\phi_m^2(z_s)$. For an ideal waveguide with M modes, by changing the source-depth M times $\{z_{s1}, z_{s2}, \ldots z_{sM}\}$, then the kernel matrix for inverting the Θ_{mj} is given by:

$$\Phi = \begin{bmatrix} \phi_1^2(z_{s1}) & \phi_2^2(z_{s1}) & \cdots & \phi_M^2(z_{s1}) \\ \phi_1^2(z_{s2}) & \phi_2^2(z_{s2}) & \cdots & \phi_M^2(z_{s2}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^2(z_{sM}) & \phi_2^2(z_{sM}) & \cdots & \phi_M^2(z_{sM}) \end{bmatrix} = \begin{bmatrix} \sin^2(y_1) & \sin^2(2y_2) & \cdots & \sin^2(My_1) \\ \sin^2(y_2) & \sin^2(2y_2) & \cdots & \sin^2(My_2) \\ \vdots & \vdots & \ddots & \vdots \\ \sin^2(y_M) & \sin^2(2y_M) & \cdots & \sin^2(My_M) \end{bmatrix} \tag{7}$$

$$y_i = (\pi z_{si} / H) \tag{8}$$

It is easy to extend the case to the Pekeris waveguide, just replace the real water-depth H by the "effective depth" H_{eff} defined as follows [4]:

$$H_{eff} = H + \Delta H = H + P/(2k_0) \tag{9}$$

where P is a bottom parameter related to bottom reflection phase shift and can be retrieved from the reverberation data[5].

By using the following two formula :

$$\sin(ny) = \sin y \times T_n(\cos y) \tag{11}$$

$$T_n(\cos y) = n\cos^{n-1} y + \frac{n(n-1)(n-2)}{3!}\cos^{n-3} y \cdot \sin^2 y + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5} y \cdot \sin^4 y + \cdots \tag{12}$$

The determinant of Φ - D_M can be expressed as :

$$D_M = \prod_{m=1}^{M} \sin^2 y_m \times H_m \tag{13}$$

where H_m is the so-called "Vandermonde determinant" defined as:

$$H_M = \begin{bmatrix} 1 & (2\cos y_1)^2 & (2\cos y_1)^4 & (2\cos y_1)^6 & \cdots & (2\cos y_1)^{2(M-1)} \\ 1 & (2\cos y_2)^2 & (2\cos y_2)^4 & (2\cos y_2)^6 & \cdots & (2\cos y_2)^{2(M-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (2\cos y_M)^2 & (2\cos y_M)^4 & (2\cos y_M)^6 & \cdots & (2\cos y_M)^{2(M-1)} \end{bmatrix} \tag{14}$$

Finally, we get

$$D_M = 2^{M(M-1)}(\prod_{i=1}^{M} x_i)(\prod_{1<i<j<M}(x_i - x_j)) \tag{15}$$

where

$$x_i = \sin^2 y_i = \sin^2(\pi z_{si} / H_{eff}) \tag{16}$$

The mapping relation of x_i and source-depth z_{si} is depicted on Fig.1.

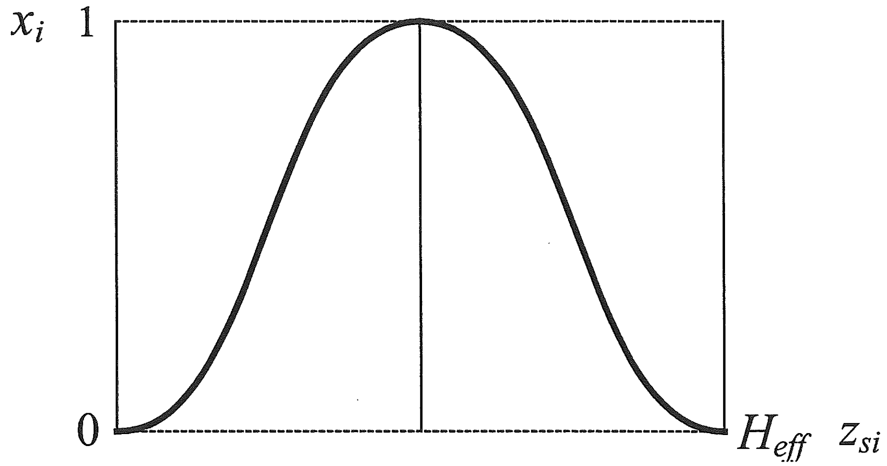

Fig.1 The mapping relation of x_i and z_i

As we can see from eq.(15), the value of D_m is a function of the source depth distribution $\{z_{si}\}$, and what we are seeking for is the optimum source depth distribution $\{z_{si}\}_{opt}$ that gives the maximum value of the D_M - $[D_M]_{Max}$.

The procedure of seeking the maximum of D_M is as follows :

Step 1: keeping the x_M as a parameter and calculating $(\partial D_M / \partial x) = 0$ for $i = 1,\ldots M\text{-}1$, to establish the relationship between x_M and x_i ($i = 1,2,.. M\text{-}1$),

Step 2: using the results from step 1 to establish D_M as a one variable function - $D_M(x_M)$, then find out the maximum Max $D_M(x_M)$ and the corresponding optimum value of x_M - $(x_M)_{\rm opt}$,

Step 3: using the relationship between x_M and x_i (i=1,2,...M-1) again to get $(x_i)_{\rm opt}$, i=1,..,M,

Step 4: using eq.(16) and eq.(8), we can finally get the optimal source depth distribution $\{z_{si}\}_{\rm opt}$.

We will skip the details and only list out the main results of $|D_M|_{Max}$ and $\{z_{si}\}_{\rm opt}$ for the case of $M = 2$ to $M = 16$. The $(x_i)_{\rm opt}$ are listed in Table 1, and the value of $|D_M|_{Max}$ are listed in Table 2.

Table 1. The optimum $\{x_i\}$ for $M = 2,3,4,\ldots.16$

$(x_i)_{opt} \setminus M$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
x_1	0.5	0.276	0.173	0.117	0.085	0.064	0.050	0.040	0.033	0.027	0.023	0.020	0.017	0.015	0.013
x_2	1.0	0.724	0.5	0.357	0.266	0.204	0.161	0.131	0.108	0.904	0.077	0.066	0.057	0.050	0.045
x_3		1.0	0.827	0.642	0.5	0.395	0.318	0.261	0.217	0.184	0.157	0.136	0.118	0.104	0.092
x_4			1.0	0.883	0.734	0.605	0.5	0.417	0.352	0.300	0.259	0.225	0.197	0.174	0.154
x_5				1.0	0.915	0.796	0.681	0.583	0.5	0.413	0.375	0.329	0.289	0.257	0.229
x_6					1.0	0.936	0.839	0.739	0.648	0.568	0.5	0.442	0.392	0.350	0.314
x_7						1.0	0.936	0.869	0.783	0.699	0.625	0.558	0.5	0.449	0.405
x_8							1.0	0.960	0.892	0.816	0.741	0.671	0.608	0.551	0.5
x_9								1.0	0.967	0.910	0.843	0.775	0.710	0.650	0.594
x_{10}									1.0	0.972	0.923	0.864	0.803	0.743	0.686
x_{11}										1.0	0.977	0.934	0.882	0.826	0.771
x_{12}											1.0	0.978	0.943	0.896	0.846
x_{13}												1.0	0.983	0.950	0.908
x_{14}													1.0	0.985	0.955
x_{15}														1.0	0.987
x_{16}															1.0

Table 2. The maximum values of D_M

M	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$(D_M)_{Max}$	1.0	1.14	1.47	2.06	3.13	5.09	8.79	16.0	30.7	61.3	127.8	276.4	619.2	1433.2	3420.8

The optimum source-depth distribution given by the $[x_{si}]_{\text{opt}}$ listed on Table 1 are the positions of the exact solution, in practical condition, we want that the requirement of the precision of the source depth distribution is not critical. As we can see from eq.(15) that under a quite tolerable condition that the matrix Φ is not singular by simply choosing the x_i are not very close to each other. We can suggest a convenient way for choosing the source-depth as a sub-optimal distribution - that is to take the equal spacing of the half water-depth:

$$\{z_{si}\}_{sub} \sim \begin{matrix} 0.5H_{eff} - (i-1)(0.5H_{eff}/M) & \text{when} & z_s < 0.5H_{eff} \\ 0.5H_{eff} + (i-1)(0.5H_{eff}/M) & \text{when} & z_s > 0.5H_{eff} \end{matrix} \qquad (17)$$

3. Numerical examples

3.1 Example 1

An acoustic point source with frequency $f = 150$ Hz in a Pekeris waveguide with water-depth $H = 50$ m, and bottom parameter c_b =1623 (m/s), $\rho_b = 1.77$, parameter $P = 10$. and $H_{eff} = 58$ m. There are 4 trapped modes. The mode function is given by :

$$\phi_m^2(z) = (2/H_{eff})\sin^2(m\pi z/H_{eff}) \qquad (18)$$

If we set up the source-depth distribution as : { 7 m, 14 m, 21 m, 28 m}, then the matrix Φ is :

$$\Phi = (0.034)\begin{bmatrix} 0.137 & 0.472 & 0.823 & 0.997 \\ 0.472 & 0.997 & 0.582 & 0.012 \\ 0.823 & 0.582 & 0.071 & 0.973 \\ 0.997 & 0.012 & 0.973 & 0.047 \end{bmatrix} \tag{19}$$

The selected source-depth is the "sub-optimal" distribution given by eq.(17), as we can see from Table 3, that they are pretty close to the "optimal" distribution given by Table 1, and the corresponding $|D_M|_{Max}$ are also very close. In this case the stability of inversion described by the "condition number" N_c is:

$$N_c = 3.1 \tag{20}$$

The condition number shown by eq.(20) means that the inversion by choosing the sub-optimal source-depth distribution given by eq.(17) is quite stable (well-posed). And it also illustrates that the requirement of the precision of the optimum position is not very critical.

Table 3. Comparison of the optimal and the sub-optimal results

	$(x_i)_{opt}$	optimal $(z_{si})_{opt}$	sub-optimal $(z_{si})_{sub}$	$(x_{si})_{sub}$
$i=1$	0.173	7.9 (m)	7.0 (m)	0.137
$i=2$	0.500	14.5 (m)	14 (m)	0.472
$i=3$	0.827	21.1 (m)	21 (m)	0.823
$i=4$	1.000	29.0 (m)	28 (m)	0.996
	$\|D_M\|_{opt} = 1.47$		$\|D_M\|_{sub} = 1.36$	

3.2 Example 2

An acoustic point source with frequency f =300 Hz in a Pekeris waveguide with water-depth H = 50 m, and the same bottom parameter as in example 1, P = 10, H_{eff}=54 m. There are 8 trapped modes. In [2], numerical simulation is conducted to compare the quality of inversion for different cases. Four cases of different source-depth distribution are completed. The four different source-depth distributions are listed in Table 4, and depicted in Fig.2. The inversion stability described by N_c are listed in Table 5.

Table 4. The source-depth distributions for different cases

Source-depth distribution $\{z_{si}\}$ (m)				
	Case A	Case B	Case C	Case D
$i=1$	26	15	22	5
$i=2$	29	18	23	11
$i=3$	32	21	24	17
$i=4$	35	24	25	23
$i=5$	38	27	26	29
$i=6$	42	30	27	35
$i=7$	45	33	28	41
$i=8$	48	36	29	47

Table 5.

Case	A	B	C	D
N_c	12	2×10^5	4×10^5	1×10^3

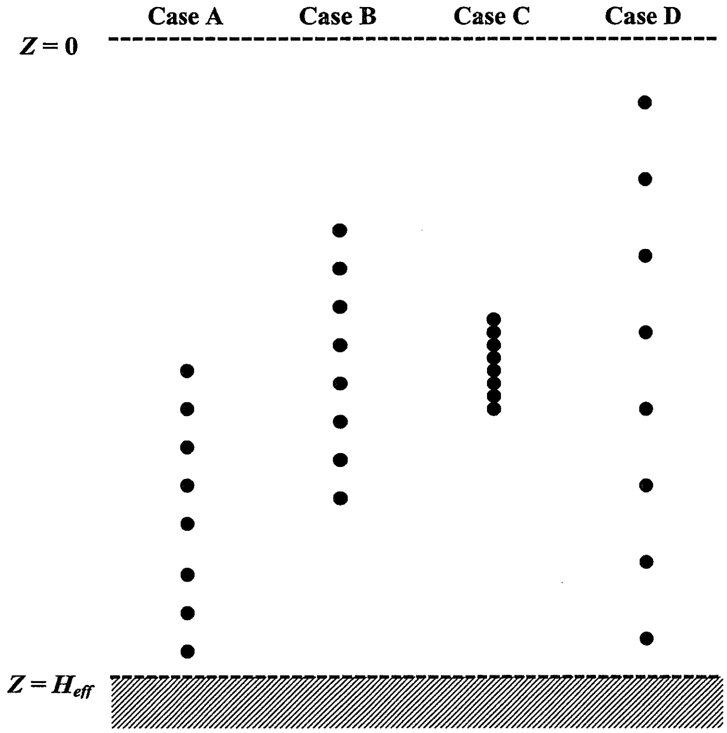

Fig.2 The source-depth distribution as listed in table 4

As we can see from Table 5, the source-depth distribution of case A can give the most stable inversion, because this source-depth distribution is quite close to the optimal source-depth distribution.

4. Summary

1) The inversion approach of the modal back-scattering matrix Θ_{mn} from the reverberation data has been proposed in [1]. The $(M\times M)$ data set applied in this approach is collected from the mode filtering (m = 1,2,...,M) combined with source-depth changing (z_{s1},z_{s2},...,z_{sM}). The kernel matrix for the inversion in a Pekeris waveguide - Φ is given by eq.(7).

2) The stability of the inversion is discussed through the calculation of the maximum of the determinant of the kernel matrix - $|D_M|_{Max.}$ The corresponding source-depth distribution is the optimal source-depth distribution – $\{z_{si}\}_{\rm opt}$. The data collected by setting the source-depth as $\{z_{si}\}_{\rm opt}$ gives the most stable inversion. The exact values of $\{z_{si}\}_{\rm opt}$ can be obtained from the mapping relationship given by eq.(16) and the $\{x_i\}_{\rm opt}$ values listed in Table 1 for M = 2 up to M = 16.

3) The requirement of the precision of the $\{z_{si}\}$ is not critical. It can conveniently to set up the desired source-depth distribution in practical situation – to take the equal spacing of the half water-depth, either the upper half or the lower half. A sub-optimal source-depth distribution is suggested by eq.(17).

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