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Application of Domain Decomposition Techniques to the Solution of Acoustic Propagation Problems

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Abstract: We study the application of the Domain Decomposition technique to the solution of the Helmholtz equation on a rectangular domain. By this technique, the domain is divided into a number of subregions. A preconditioner is derived which can be efficiently applied by solving independent problems on the subregions and a system for the unknowns on the separator set, given by points at the interfaces between subdomains. The method is particularly attractive because of its suitability to implementation on modern parallel supercomputers. Also, the method has potential for great modularity, such as the possibility of combining more accurate solvers in the near field with more efficient approximations in the far field, where backscattering is no longer expected. Some of the issues we need to deal with are the fact that the discretized differential operator is indefinite, and the correct handling of radiation conditions.

1. INTRODUCTION

In this paper, we consider the numerical solution of the Helmholtz equation

$$\Delta u + k^2 u = 0 \ \ in \ \ \Omega, \tag{1.1}$$

which describes wave propagation phenomena in underwater acoustics. In the present work we concentrate on two dimensional problems, and assume that Ω is a rectangle. We assume Dirichlet or Neumann boundary conditions on three of the boundaries and a finite approximation to the radiation condition at the far wall.

A finite-difference or finite element discretization leads to a linear system

$$AU = f,$$

where U is the discrete solution vector. The matrix A has real entries, except for those elements corresponding to gridpoints located at the far wall. Also, A is indefinite, and in general not symmetric.

For the cases when cartesian coordinates are used and the coefficient k^2 is constant or separable, fast direct solvers can be applied to solve (1.2) in $O(nm \log n)$ time, where $n \times m$ is the size of the finite difference gird imposed on Ω. For general variable k^2, the system must be solved iteratively. Since A is indefinite and not symmetric, standard iterative method are not applicable. Instead, conjugate gradients can be applied to the solution of the normal equations $A^TAU = A^Tf$, or the system (1.2) can be solved by other Krylov space methods suitable for indefinite systems, such s the generalized minimal residual method GMRES [5]. At the i-th step, GMRES produces a new approximate solution which minimizes the Euclidean norm of the residual on a space of dimension $i + 1$. The efficiency of the method depends upon finding a good preconditioner M, which approximates the inverse of A (thus keeping the iteration count low), and at the same time can be applied at low computational cost.

In this paper we propose preconditioners for the matrix A derived from domain decomposition techniques, which consist on approximating the solution of an elliptic problem by solving simpler problems on subregions of Ω, and at the interfaces between subregions.

In the next section, we describe the method domain decomposition as a direct solver, in particular as it applies to the Helmholtz equation on a rectangular domain, for the case where the coefficient k^2 is piece-wise constant. In section 3, we discussion the introduction of wall boundary conditions. In section 4, we consider the domain decomposition solver as a preconditioner for more general problems, and discuss its efficiency. We also apply this preconditioner to a model problem as an example.

2. DOMAIN DECOMPOSTION

The terms *Domain Decomposition* and *Substructuring* refer to techniques for solving partial differential equations by first decomposing the domain into smaller regions or subdomains and then reducing the solution of the original problem to solving problems on these subdomains. The decomposition is sometimes motivated by the physics of the problem; in other cases, the domain is decomposed in order to simplify the computation of the solution. The i dea is well suited to the numerical solution of PDE's in a parallel environment, since the subproblems can be solved independently and communication is limited to the boundaries of the subdomains.

The general concept of domain decomposition is not new. It can be traced to Schwarz's alternating procedure, which proves the existence of a solution for a boundary value problem

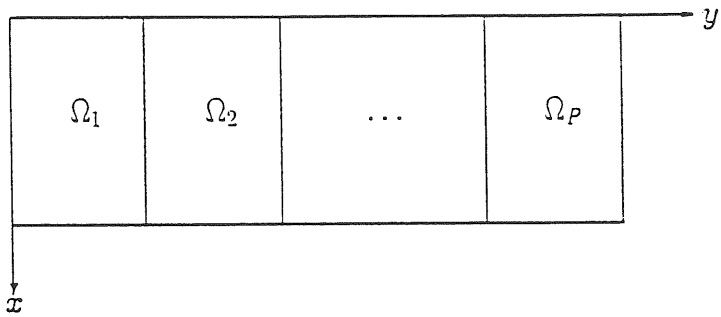

Figure 1: Decomposition of rectangular domain Ω

by an iteration which involves overlapping subdomains [6]. The rate of convergence of this procedure depends on the amount of overlapping: more overlapping implies faster convergence, but more work per iteration, since the work is duplicated on the overlapping region.

The methods considered in this paper apply to non-overlapping subdomains. The basic idea is to reduce the differential operator on the whole domain to an operator on the interfaces. After solving for the interface values, the solution at the interior of the subdomains can be computed by solving independent problems with the computed interface values as boundary conditions.

2.1 Description of the Method

The domain is discretized and partitioned into several subregions; then, by applying block elimination to the discretized equations, a system is derived for the unknowns on the interfaces between subregions. Once this system is solved and the solution is known at the interfaces, the original problem is decoupled and the solution at the interior of the subdomains can be found by solving independent problems on the subdomains. Forming the right hand side for the interface system requires the solution of independent elliptic problems on the subdomains. For constant coefficient problems on regular domains, fast direct methods can be applied to the solution of this system. Such is not the case, however, for more general operators on irregular domains. For efficiency reasons, the system is then solved by iterative methods, such as the preconditioned Krylov methods, with preconditioners that are derived from approximations to the subdomain operators and to the interface system. We discuss these preconditioners in section 4.

In order to illustrate the domain decomposition method, consider the problem:

$$Lu = 0 \quad \text{in} \quad \Omega \ , \tag{2.1}$$

subject to certain boundary conditions on $\partial\Omega$. L is a linear elliptic operator and Ω is the domain in Fig. 1, which can be decomposed into subdomains $\Omega_1, \ldots, \Omega_p$, separated by the interfaces Γ_i.

By applying a five-point discretization to (2.1), a linear system

$$AU = f \tag{2.2}$$

is derived, where U represents the discrete solution vector.

Let Γ be the union of all separators between subdomains. If the vectors U and f are reordered such that the points in the interior of the subdomains are numbered first, then

(2.2) can be written in block for as:

$$\begin{pmatrix} A_\Omega & P \\ P^T & A_\Gamma \end{pmatrix} \begin{pmatrix} U_\Omega \\ U_\Gamma \end{pmatrix} = \begin{pmatrix} f_\Omega \\ f_\Gamma \end{pmatrix} \quad . \tag{2.3}$$

Since there is no coupling between the interior points of two different subdomains, the submatrix A_Ω is block diagonal:

$$A_\Omega = \begin{pmatrix} A_{\Omega_1} & & & \\ & A_{\Omega_2} & & \\ & & \ddots & \\ & & & A_{\Omega_k} \end{pmatrix} \quad ,$$

where each block A_{Ω_i} represents the discretization of the differential operator on the interior of the i-th subdomain. A_Γ represents the restriction of the operator to the interfaces and P represents the coupling between subdomains and interfaces.

Using the following block decomposition of the matrix A,

$$A = \begin{pmatrix} A_\Omega & 0 \\ P^T & S \end{pmatrix} \begin{pmatrix} I & A_\Omega^{-1} P \\ 0 & I \end{pmatrix} \quad , \tag{2.4}$$

where the interface matrix S is the Schur complement of A_Γ in A, i.e.:

$$S = A_\Gamma - P^T A_\Omega^{-1} P \quad , \tag{2.5}$$

system (2.3) can be solved by block-Gaussian elimination as follows:

ALGORITHM DD:
Step 1: Compute $g = f_\Gamma - P^T A_\Omega^{-1} f_\Omega$.
Step 2: Solve the interface system $SU_\Gamma = g$.
Step 3: Solve for interior subdomain points: $A_\Omega U_\Omega = f_\Omega - PU_\Gamma$.

Note that, except for Step 2, the algorithm only requires the solution of problems with A_Ω, which corresponds to solving independent problems on each Ω_i.

The computation of the right hand side g requires the solution of one problem on each subdomain and a sparse matrix-vector product.

The Schur complement S has the following block-tridiagonal form:

$$S = \begin{pmatrix} H_1 & B_2 & & \\ B_2 & H_2 & \ddots & \\ & \ddots & \ddots & B_{p-1} \\ & & B_{p-1} & H_{p-1} \end{pmatrix} \quad . \tag{2.6}$$

The explicit computation of the matrix S is expensive. Also, in most cases, the solution of $SU_\Gamma = g$ by Gaussian elimination would exceed the complexity of solving the subdomain problems. However, for piece-wise constant coefficient problems, the blocks H_i and B_i are diagonalizable by Fourier modes, i.e.

$$W^T H_i W = \Lambda_i = \text{ diag}(\lambda_{i1}, \ldots, \lambda_{in}) \tag{2.7}$$

and

$$W^T B_i W = D_i = \text{ diag}(\delta_{i1}, \ldots, \delta_{in}) \quad , \tag{2.8}$$

where W represents a matrix of Fourier modes (e.g. sine modes for the case of Dirichlet boundary conditions) and the eigenvalues λ and δ can be expressed in terms of the coefficients of the differential operator and the size of the subdomains.

The interface system can then be rapidly solved by *matrix decomposition* [2], by applying fast Fourier transforms and solving n (where n is the number of grid-points on each interface) decoupled tridiagonal systems of dimension $p-1$. For a full description of the method and a derivation of the expressions for λ and δ for several operators and domain splittings, see [4].

We now concentrate our attention on the Helmholtz problem on a rectangular domain. Consider first the Helmholtz operator in cartesian coordinates:

$$Lu \equiv u_{xx} + u_{yy} + k^2 u$$

with Dirichlet boundary conditions on $\partial\Omega$. Assume that the coefficient k takes constant values k_i inside each subdomain Ω_i. Let $n \times m_i$ be the dimensions of the grid inside subdomain Ω_i, with $m+1 = \sum(m_i+1)$. We also assume that the problem has been scaled, so that $\Omega = [0,1] \times [0, Y_W]$. Let $h = \frac{1}{n+1}$. Each subdomain block A_{Ω_i} has the following tridiagonal form:

$$A_{\Omega_i} = \begin{pmatrix} T_i & I & & \\ I & T_i & \ddots & \\ & \ddots & \ddots & I \\ & & I & T_i \end{pmatrix}$$

with m_i blocks, where $T_i = \text{tridiag}(1, -4 + k_i^2 h^2, 1)$. T_i, as well as all diagonal blocks in A_Γ, are diagonalizable by Fourier modes, i.e.

$$T_i = W \ \text{diag}(t_{i1}, \ldots, t_{in}) W^T$$

and

$$A_{\Gamma_i} = W \ \text{diag}(\beta_{i1}, \ldots, \beta_{in}) \ W^T \quad ,$$

where $t_{ij} = -2 - 4\sin^2 j\frac{\pi}{2}h + k_i^2 h^2$, $\beta ij = -2 - 4\sin^2 j\frac{\pi}{2}h + \frac{k_i^2 + k_{i+1}^2}{2}h^2$ and the elements of W are given by:

$$w_{ij} = \sqrt{2h} \sin ijh\pi \quad . \tag{2.9}$$

Then, we can show that the blocks of the Schur complement S can be diagonalized as follows:

$$W^T H_i W = \Lambda_i = \ \text{diag}(\lambda_{i1}, \ldots, \lambda_{in})$$

$$W^T B_i W = D_i = \ \text{diag}(\delta_{i1}, \ldots, \delta_{in}) \quad .$$

Let $s_{ij} = k_i^2 h^2 - 4\sin^2 j\frac{\pi}{2}h$. Then, the eigenvalues λ_{ij} of H_i are given by

$$\lambda_{ij} = \phi_{ij} + \phi_{i+1,j} \quad ,$$

where

$$\phi_{ij} = -\sqrt{\frac{s_{ij}^2}{4} - s_{ij}} \ \frac{1 + \gamma_j^{m_i+1}}{1 - \gamma_j^{m_i+1}} \quad \text{if} \quad |1 - \frac{s_{ij}}{2}| > 1 \tag{2.10}$$

with $\gamma_{ij} = \left(1 - \frac{s_{ij}}{2} - \sqrt{\frac{s_{ij}^2}{4} - s_{ij}}\right)^2$, and

$$\phi_{ij} = -\frac{\sin\theta_{ij}}{\tan(m_i+1)\theta_{ij}} \quad \text{if} \quad |1 - \frac{s_{ij}}{2}| < 1 \quad , \tag{2.11}$$

with $\cos\theta_{ij} = 1 - \frac{s_{ij}}{2}$. The eigenvalues of B_i are given by

$$\delta_{ij} = 2\sqrt{\frac{s_{ij}^2}{4} - s_{ij}}\ \frac{\sqrt{\gamma_{ij}^{m_i+1}}}{1-\gamma_{ij}^{m_i+1}} \quad \text{if} \quad |1 - \frac{s_{ij}}{2}| > 1 \tag{2.12}$$

and

$$\delta_{ij} = \frac{\sin\theta_{ij}}{\sin(m_i+1)\theta_{ij}} \quad \text{if} \quad |1 - \frac{s_{ij}}{2}| < 1. \tag{2.13}$$

When the Dirichlet boundary conditions at the bottom are replaced by Neumann boundary conditions, the expressions for λ_{ij} and δ_{ij} are similar, and the eigenvectors are given by a matrix W, whose elements are

$$w_{ij} = \sqrt{2h}\sin(2i-1)jh\frac{\pi}{2} \quad .$$

where $h = \frac{1}{n}$.

3. RADIATION BOUNDARY CONDITIONS

When a radiation boundary conditions is given at a far wall, the matrix A becomes complex. However, the coupling between real part and imaginary part of the solution is reduced to the last blocks of the matrix.

Consider an approximation of the radiation condition at the far wall, given by:

$$\frac{\partial u}{\partial y} = Qu \quad \text{at} \quad y = Y_W \quad , \tag{3.1}$$

where Q is a complex linear operator.

By (3.1), we get an expression for the solution vector at $Y_W + h$ in terms of the solution at Y_W:

$$U_{m+1} = e^{hQ}U_m \tag{3.2}$$

Therefore, the matrix A in this case takes the same form as for Dirichlet boundary conditions at the wall, except for the last block, which becomes $T_p + e^{hQ}$. Although an explicit computation of the elements of e^{hQ} would not be practical, the product of such operator with a vector of size n (which is all we need for the application of an iterative method such as GMRES), can be approximated efficiently by a finite difference scheme.

By including the gridpoints on the far wall as part of the interface, all submatrices A_{Ω_i} are real, therefore each subdomain problem can be solved for the real and imaginary parts of the solution as two decoupled real linear systems. The effect of including the wall gridpoints with the other interior interfaces, is that the expression for the eigenvalues λ_{ij} for $i = p$ is different from the others.

4. DD PRECONDITIONER: AN EXAMPLE

As we mentioned before, for the case of general variable k^2, the subdomain problems cannot be solved by fast direct methods, and the eigenvalues of the interface matrix are not known. The system $AU = f$ can be solved by iterative methods such as GMRES, with a domain decomposed solver as a preconditioner, where the coefficient k^2 is replaced by piece-wise constant or separable approximations on the subdomains. The convergence rate will depend on how well the coefficient is approximated. For a more detailed discussion of this issue, see [4].

As an example, we apply a domain decomposition preconditioner to the solution of the equation

$$\begin{aligned} u_{xx} + u_{yy} + k^2 u &= 0 \quad \text{in} \quad \Omega \\ u(x,0) &= g_0(x) \\ u(0,y) &= 0 \\ u(1,y) &= 0 \\ u(x,Y_W) &= g_W(x) \end{aligned}$$

where Ω is the unit square and k^2 is given by the following expression from [1]:

$$k^2(x,y) = k_0^2 \left[a + \frac{1-a}{\cosh^2 C_1(x-\frac{1}{2})} + b \tanh C_2(y-q) \right]^2 ,$$

with $a = 0.93652$, $b = -0.121$, $C_1 = 20$, $C_2 = 50$, and $q = 0.25$.

In this case, there are two decoupled linear systems for the real and imaginary parts of the solution. We divided the domain into two equal subdomains and applied a GMRES solver with a domain decomposed preconditioner. Inside each subdomain, the coefficient k^2 is first approximated by a constant and second, by a separable approximation. We remark that in this later case, since k^2 is already separable, the subdomain problems are solved exactly, while the interface system is replaced by an approximation.

In Fig. 2 we show the reduction of the norm of the residual at each iteration for: (A) a piece-wise constant coefficient approximation and (B) a piece-wise separable approximation. For the sake of comparison, (C) shows the residual reduction when the Laplacian operator (i.e. $k^2 = 0$) is used as a preconditioner, showing a much slower rate of convergence. The multigrid preconditioner used in [3] approximates the solution of a Laplacian problem at each iteration. Our results show that, by considering other approximations to k^2, the rate of convergence is significantly improved, while the preconditioner can still be efficiently applied by combining the techniques of domain decomposition and fast solvers in the subdomains.

5. CONCLUSION

We introduced a class of preconditioners for the solution of the Helmholtz equation on a rectangular domain, derived from domain decomposition techniques. These preconditioners can be efficiently applied by solving independent problems on subregions and a system for the unknowns on the separator set, and they are especially suited for parallel implementation. Our preliminary tests show satisfactory convergence rate for a variable coefficient model problem. Further testing and research will be necessary in order to evaluate the performance of these preconditioners for more general problems and boundary conditions.

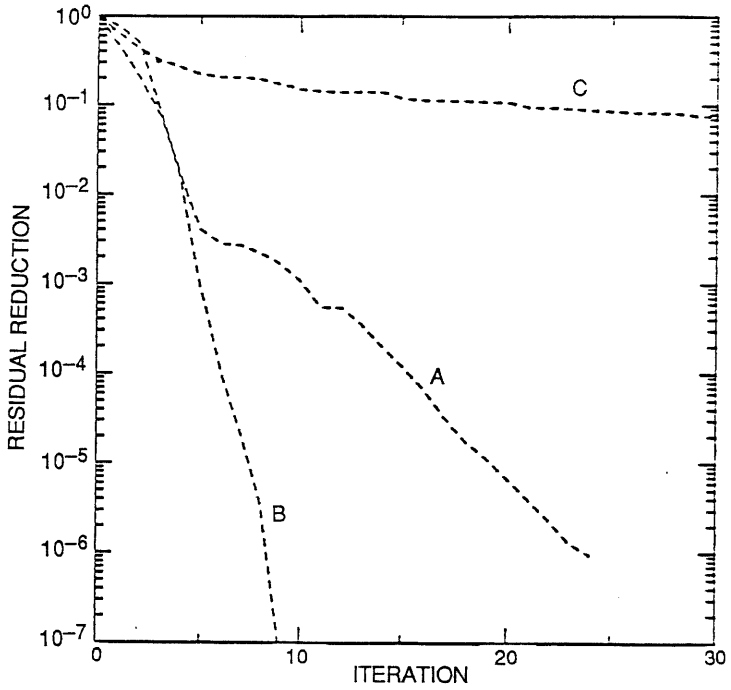

Figure 2. Residual reduction for model problem. GMRES solver with: DD preconditioner with piece-wise constant approximation (A), DD preconditioner with piece-wise separable approximation (B), and preconditioned by the Laplacian operator (C).

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