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Comparison of Kirchhoff and T-Matrix Solutions for Sound Scattering from Spheroids

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Abstract: Based on its usual derivation in terms of a local plane-wave approximation, the Kirchhoff method for acoustic scattering often performs better than one would expect, especially in the forward direction. A derivation based on the on-surface radiation condition (OSRC) method provides insight into the performance of the Kirchhoff method for diffracted energy. Kirchhoff solutions are compared with T-matrix solutions for high-frequency acoustic scattering from rigid spheroids. The Kirchhoff method does not account for creeping waves, which are important for end-on incidence especially for spheroids of large aspect ratio. The Kirchhoff method performs well if creeping waves are not excited.

1. INTRODUCTION

The standard derivation of the Kirchhoff method for high-frequency acoustic scattering is based on a local plane-wave approximation on the surface of the scatterer [1]. The surprisingly good performance of the Kirchhoff method seems to be inconsistent with the geometrical assumptions made in the derivation: accuracy is often greatest for forward scattering, which involves diffracted energy rather than specularly-reflected energy. In this paper, we present a derivation based on the on-surface radiation condition (OSRC) method [2], which involves a parabolic approximation [3,4] on the surface of the scatterer. Parabolic approximations account for diffraction, are valid when energy propagates within 90 degrees of a preferred direction, and are valid for lower frequencies than ray methods. Higher-order parabolic approximations are very accurate for propagation [5,6] and scattering [7].

The OSRC method was motivated by the results of numerical experiments involving a finite-difference solution for the scattered field using a computational grid surrounding the scatterer and a radiation boundary condition (RBC) to truncate the grid. The accuracy of this solution depends on the accuracy of the RBC and the distance from the surface of the scatterer to the boundary of the grid (the accuracy increases as the boundary moves toward infinity where normal incidence occurs). It was observed that the latter condition may be relaxed in many cases, and the OSRC method was obtained by allowing the grid to collapse onto the surface of the scatterer. The scattered field is obtained as an integral over the surface of the scatterer; the integrand is obtained directly for soft scatterers but requires the solution of a differential equation on the surface of the scatterer for rigid scatterers.

The OSRC method reduces to the Kirchhoff method in the limit of high frequency and an arbitrarily-accurate RBC. Thus the Kirchhoff solution is equivalent to the finite-difference solution in the limit of high frequency and an arbitrarily-accurate RBC placed arbitrarily close to the surface of the scatterer. With this interpretation, there is no reason to expect the Kirchhoff method to be accurate only for specular scattering. Kirchhoff solutions are compared with reference solutions generated with the T-matrix method [8,9] for acoustic scattering from rigid spheroids. The Kirchhoff method performs well for rigid spheroids except when creeping waves are important.

2. A DERIVATION BASED ON THE OSRC METHOD

The surface $\partial\Gamma$ of the rigid spheroid Γ is defined in terms of Cartesian coördinates by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1 , \tag{2.1}$$

where $b \geq a$. In the region exterior to Γ, the acoustic pressure field p satisfies the Helmholtz equation

$$\left(\nabla^2 + k^2\right) p = 0 , \tag{2.2}$$

where k is the acoustic wavenumber. The incident field p_i, which satisfies (2.2), is the plane wave

$$p_i = \exp\left(i\vec{k} \cdot \vec{x}\right) , \tag{2.3}$$

where $\vec{k}$ is the wavenumber vector. The scattered field $p_s = p - p_i$ satisfies (2.2) and has the following representation in terms of the free-space Green's function G:

$$p_s = \int_{\partial\Gamma} p \frac{\partial G}{\partial n} dA , \tag{2.4}$$

$$G\left(\vec{x}, \vec{\xi}\right) = \frac{\exp\left(ik\left|\vec{x} - \vec{\xi}\right|\right)}{4\pi\left|\vec{x} - \vec{\xi}\right|} , \tag{2.5}$$

where $\partial/\partial n = \hat{n} \cdot \nabla$ and $\hat{n}$ is the outward unit normal vector on $\partial\Gamma$. In the standard derivation [1], the Kirchhoff method is obtained from (2.4) by assuming that $\partial\Gamma$ appears planar to the incident field and substituting the scattered field $p_s = p_i$ for a rigid plane to obtain

$$p_s \cong 2 \int_{\text{lit } \partial\Gamma} p_i \frac{\partial G}{\partial n} dA \,, \tag{2.6}$$

where $\vec{k} \cdot \hat{n} < 0$ in the lit region of $\partial\Gamma$.

The OSRC method is based on representing the normal derivative of p_s on $\partial\Gamma$ with a parabolic approximation. With this approach, it is relatively difficult to account for curvature terms. In the limit $ka >> 1$, we may neglect these terms to obtain the approximation

$$\frac{\partial p_s}{\partial n} = i\sqrt{k^2 + \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}}\; p_s \,, \tag{2.7}$$

where u and v are orthogonal coordinates on $\partial\Gamma$. Off the curve bounding the lit region, we also obtain the approximation

$$\frac{\partial p_i}{\partial n} = i\sigma\sqrt{k^2 + \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}}\; p_i \,, \tag{2.8}$$

where $\sigma = -1$ in the lit region and $\sigma = 1$ in the dark region. Since $\partial p/\partial n = 0$ on $\partial\Gamma$, it follows from (2.7) and (2.8) that

$$\sqrt{k^2 + \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}}\; (p_s + \sigma p_i) = 0 \,. \tag{2.9}$$

From (2.9), we conclude that $p_s \cong -\sigma p_i$ on $\partial\Gamma$ and obtain the Kirchoff solution,

$$p_s \cong \int_{\partial\Gamma} (1-\sigma) p_i \frac{\partial G}{\partial n} dA \;= 2 \int_{\text{lit } \partial\Gamma} p_i \frac{\partial G}{\partial n} dA \,. \tag{2.10}$$

3. COMPARISON WITH T-MATRIX SOLUTIONS

We use T-matrix solutions to test the accuracy of the Kirchhoff method for various scatterers and angles of incidence. We first consider spheres for the cases $ka = 20$ and $ka = 50$. The Kirchhoff and T-matrix solutions appear in Figure 1. The agreement is fairly good for $ka = 20$ and very good for $ka = 50$. Based on the usual derivation of the Kirchhoff method, it is surprising that the agreement is best in the forward direction.

We observe from Figure 1 that the T-matrix amplitude is nearly constant away from the forward direction. Thus the stationary-phase approximation of the Kirchhoff integral [1], which also predicts this behavior, is more accurate than the exact Kirchhoff integral, which oscillates slightly away from the forward direction. This unusual behavior is apparently due to the contribution to the Kirchhoff integral by the boundary of the lit and dark regions. This contribution, which is estimated using integration by parts [1], vanishes for soft scatterers because the Kirchhoff integrand vanishes on the boundary.

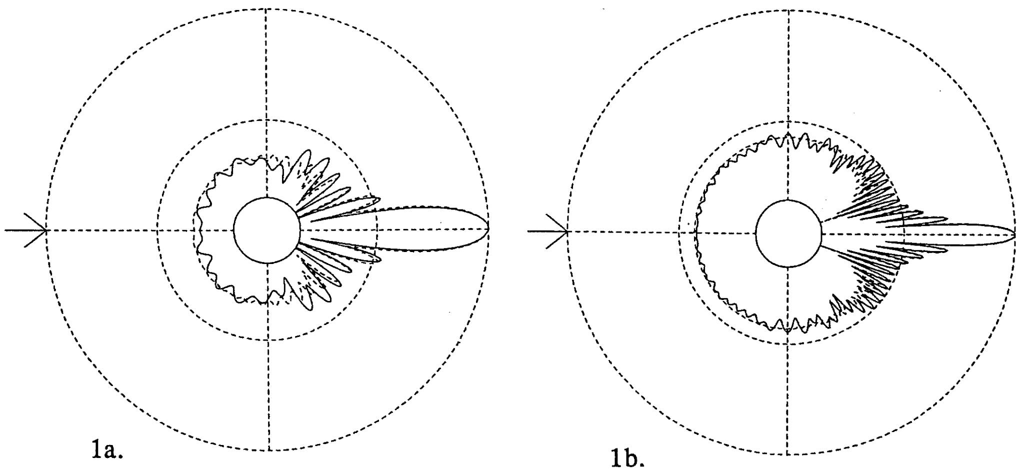

Fig. 1. Kirchhoff solutions (solid curves) and T-matrix solutions (dashed curves) for rigid spheres for (a) $ka = kb = 20$ and (b) $ka = kb = 50$. The arrow shows the direction of the incident plane wave.

Figure 2 contains results for end-on incidence for three spheroids. One would not expect the Kirchhoff method to perform well for these cases because creeping waves are highly excited for end-on incidence; the degree of excitation increases as the aspect ratio b/a increases. We observe from Figure 2 that the Kirchhoff solution degrades from fair for $b/a = 2$ to poor for $b/a = 10$. Figure 3 contains results for broadside incidence. The Kirchhoff method performs well for all of the spheroids. As one would expect, the performance improves with frequency. In contrast to what one would expect from the standard derivation of the Kirchhoff method, the performance is best in the forward direction. The results for oblique incidence appear in Figure 4. The performance of the Kirchhoff method is about the same as for broadside incidence.

4. CONCLUSION

A derivation of the Kirchhoff method based on the OSRC method provides insight into the performance of the Kirchhoff method for diffracted energy. Numerical solutions generated with the Kirchhoff method were compared with exact solutions generated with the T-matrix method. The Kirchhoff method performs well for broadside and oblique incidence. The Kirchhoff method does not perform well for end-on incidence because creeping waves are highly excited. The Kirchhoff method performs best for forward scattering. For some cases, the scattered field obtained from the stationary phase approximation of the Kirchhoff integral is more accurate away from the forward direction than the exact Kirchhoff integral.

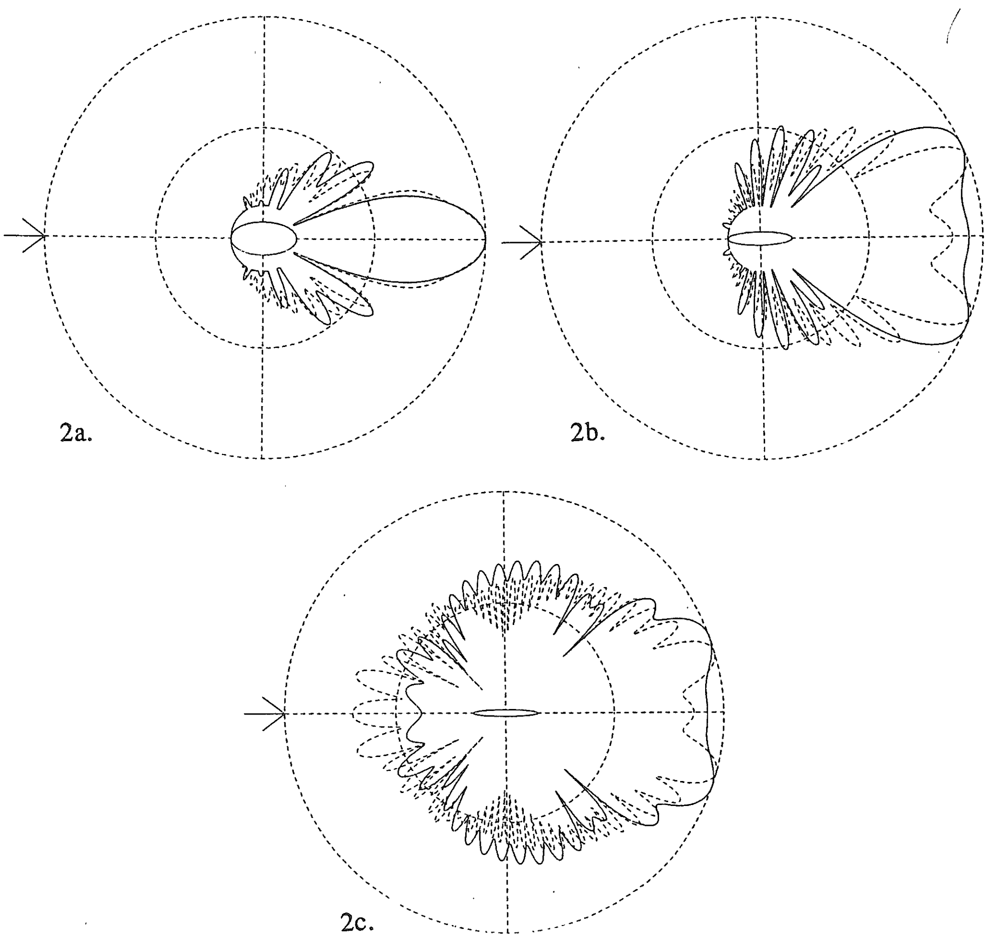

Fig. 2. Kirchhoff solutions (solid curves) and T-matrix solutions (dashed curves) for rigid spheroids for (a) $kb = 20$ and $ka = 10$, (b) $kb = 25$ and $ka = 5$, and (c) $kb = 50$ and $ka = 5$. The arrow shows the direction of the end-on incident plane wave.

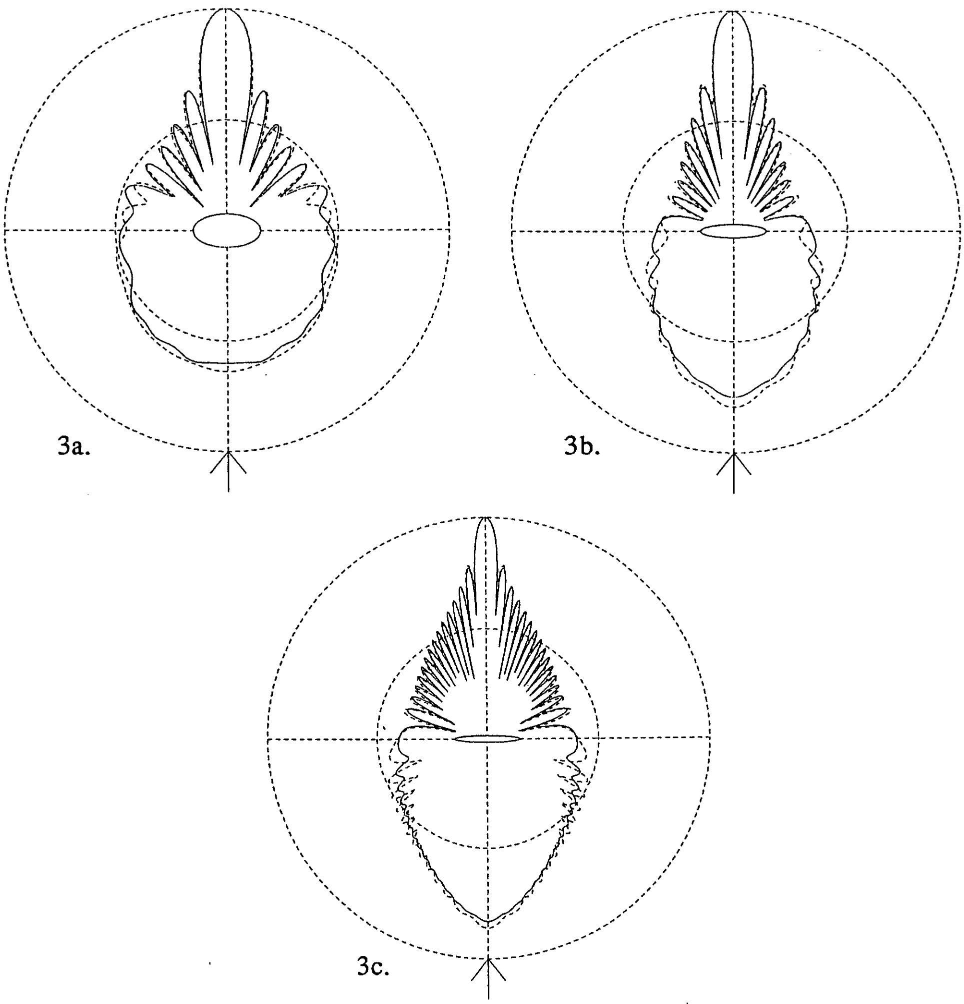

Fig. 3. Kirchhoff solutions (solid curves) and T-matrix solutions (dashed curves) for rigid spheroids for (a) $kb = 20$ and $ka = 10$, (b) $kb = 25$ and $ka = 5$, and (c) $kb = 50$ and $ka = 5$.
The arrow shows the direction of the broadside incident plane wave.

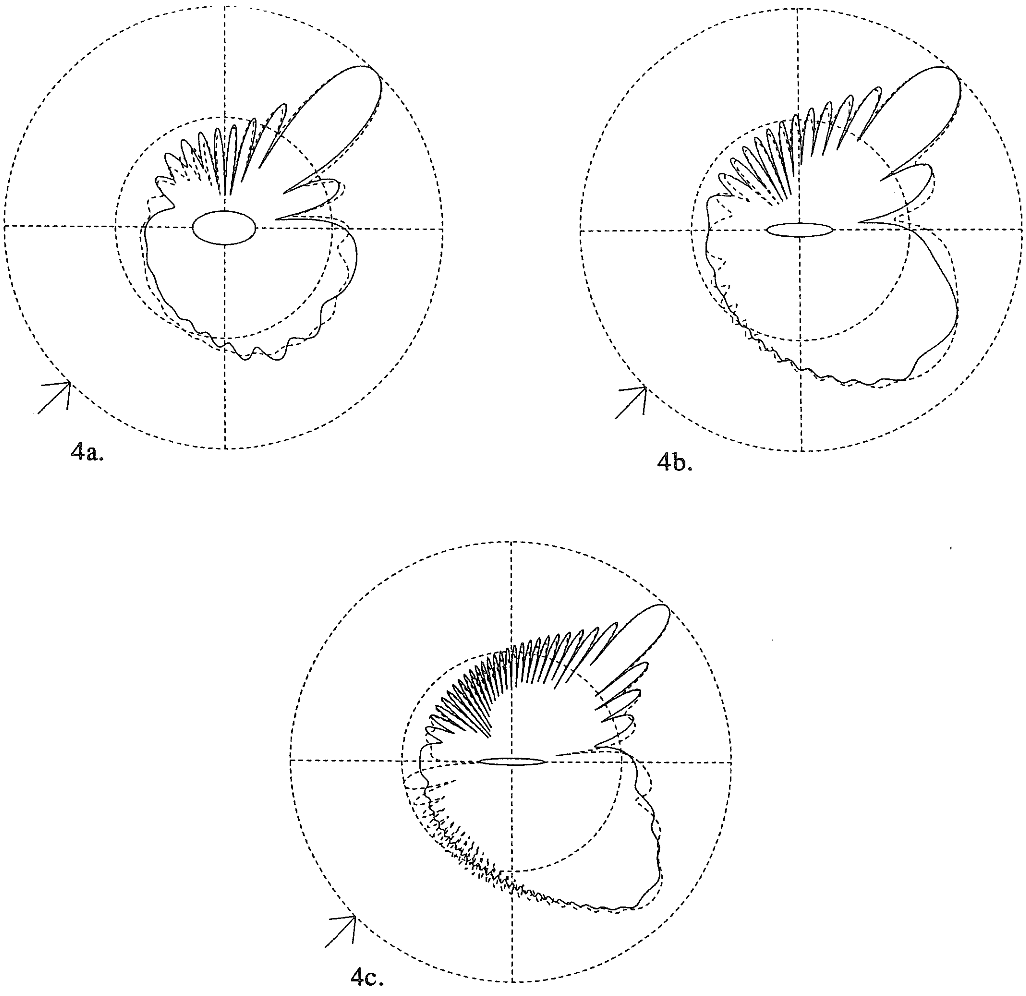

Fig. 4. Kirchhoff solutions (solid curves) and T-matrix solutions (dashed curves) for rigid spheroids for (a) $kb = 20$ and $ka = 10$, (b) $kb = 25$ and $ka = 5$, and (c) $kb = 50$ and $ka = 5$. The arrow shows the direction of the obliquely incident plane wave.

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